Chapter 1

The Definite Integral

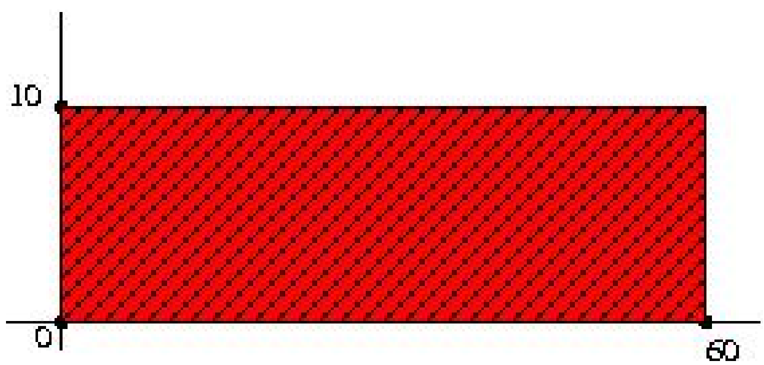
1.1 Calculating Distance - an Example

Problem 1.1.1 *A car travels in a straight line for one minute, at a constant speed of* $10\mathrm{m/s}$*. How far has the car travelled in this minute?*

SOLUTION: The car is travelling for 60 seconds, and covering 10 metres in each second, so in total it covers $60 \times 10 = 600$ metres.
That wasn't very hard.

An easy example like this one can be a starting point for studying more complicated problems. What makes this example easy is that the car's speed is not changing so all we have to do is multiply the distance covered in one second by the number of seconds. Note that we can interpret the answer graphically as follows.

Suppose we draw a graph of the car's speed against time, where the x-axis is labelled in seconds and the y-axis in m/s. The graph is just the horizontal line $y = 10$ of course.



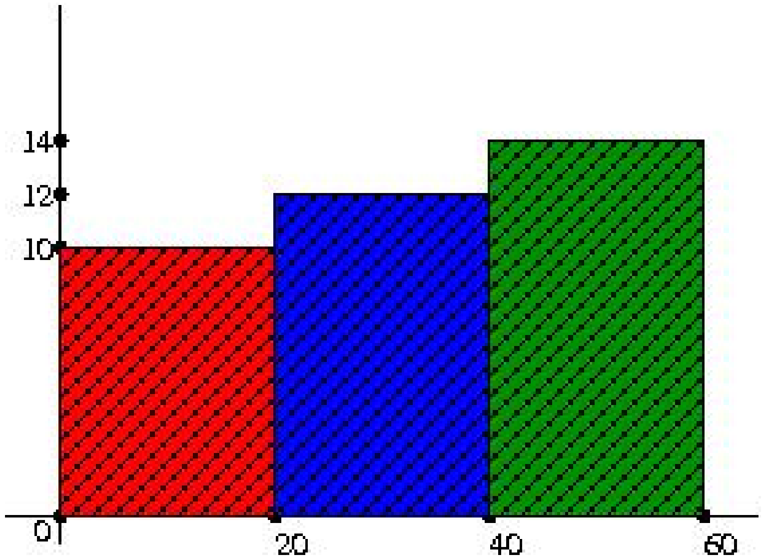
We can label the time when we start observing the car's motion as $t = 0$ and the time when we stop as $t = 60$. Note then that the total distance travelled – 600m – is the area enclosed under the graph, between the x-axis, the horizontal line $y = 10$, and the vertical lines $x = 0$ (or time $t = 0$) and $x = 60$ marking the beginning and end of the period of observation. This is no coincidence; if we divide this rectangular region into vertical strips of width 1, one for each second, what we get are 60 vertical strips of width 1 and height 10, each accounting for 10 units of area, and each accounting for 10 metres of travel.
The next problem is a slightly harder example of the same type.

Problem 1.1.2 *Again the car travels in one direction for one minute. This time it travels at* $10\mathrm{m/s}$ *for the first 20 seconds, at* $12\mathrm{m/s}$ *for the next 20 seconds, and at* $14\mathrm{m/s}$ *for the last 20 seconds.*
What is the total distance travelled?

SOLUTION: This is not much harder really (although it may be a physically unrealistic problem - why?). This time, the car covers

- $20 \times 10 = 200$ metres in the first 20 seconds
- $20 \times 12 = 240$ metres in the next 20 seconds and
- $20 \times 14 = 280$ metres in the last 20 seconds,

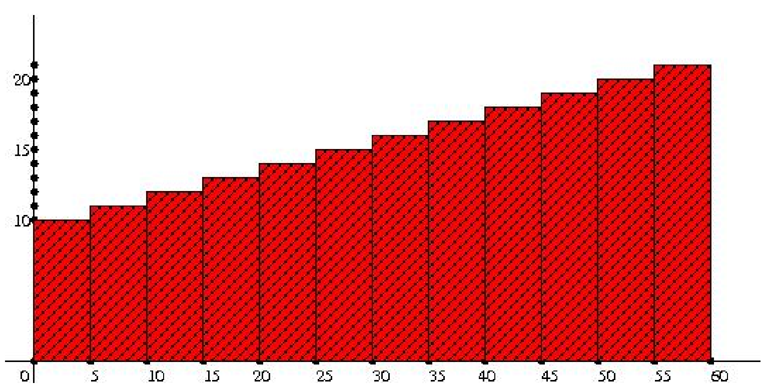
so the total distance is 720 metres.



Once again the total distance travelled is the area of the region enclosed between the lines $x = 0$, $x = 60$, the y-axis and the graph showing speed against time. The region whose area represents the distance travelled is the union of three rectangles, all of width 20, and of heights 10, 12 and 14.

Problem 1.1.3 *Same set up, but this time the car's speed is* $10\mathrm{m/s}$ *for the first 5 seconds,* $11\mathrm{m/s}$ *for the next 5, and so on, increasing by* $1\mathrm{m/s}$ *every five seconds so that the speed is 21* $\mathrm{m/s}$ *for the last five seconds. Again the problem is to calculate the total distance travelled in metres.*

The answer is left as an exercise, but this time the distance is the area indicated below.

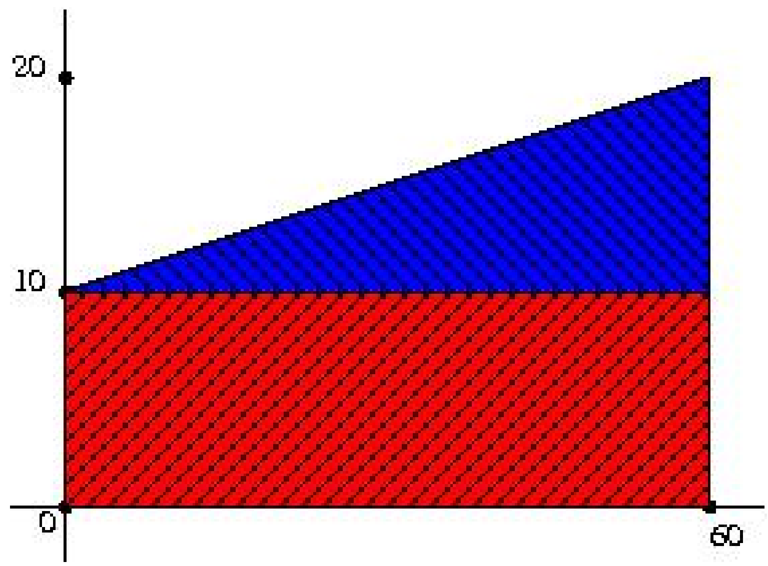


Problem 1.1.4 *Again our car is travelling in one direction for one minute, but this time its speed increases at a constant rate from* $10\mathrm{m/s}$ *at the start of the minute, to* $20\mathrm{m/s}$ *at the end. What is the distance travelled?*

NOTE: This is a more realistic problem, in which the speed is increasing at a constant rate. This constant acceleration would apply for example in the case of an object falling freely under gravity.

SOLUTION: This is a different problem from the others. Because the speed is varying *all the time* this problem cannot be solved by just multiplying the speed by the time or by a combination of such steps as in Problems 1.1.1 and 1.1.2.

The following picture shows the graph of the speed against time.



If the total distance travelled is represented in this example, as in the others, by the area under the speed graph between $t = 0$ and $t = 60$, then we can observe that it's the area of a region consisting of a rectangle of width 60 and height 10, and a triangle of width 60 and perpendicular height 10. Thus the total distance travelled is given by

$$(60 \times 10) + \frac{1}{2}(60 \times 10) = 900\mathrm{m}.$$

QUESTION: Should we believe this answer? Just because the distance is given by the area under the graph when the speed is constant, how do we know the same applies in cases where the speed is varying continuously? Here is an argument that might justify this claim.

In Problem 1.1.4, the speed increases steadily from $10\mathrm{m/s}$ to $20\mathrm{m/S}$ over the 60 seconds. We want to calculate the distance travelled.

We can approximate this distance as follows.

- Suppose we divide the one minute into 30 two-second intervals.
- At the start of the first two-second interval, the car is travelling at $10\mathrm{m/s}$. We make the *simplifying assumption* that the car travels at $10\mathrm{m/s}$ throughout the first two seconds, thereby covering $20\mathrm{m}$ in the first two seconds. Note that this actually underestimates the true distance travelled in the first second, because in fact the speed is increasing from $20\mathrm{m/s}$ during these two seconds.
- At the start of the second two-second interval, the car has completed one-thirtieth of its acceleration from $10\mathrm{m/s}$ to $20\mathrm{m/s}$, so its speed is

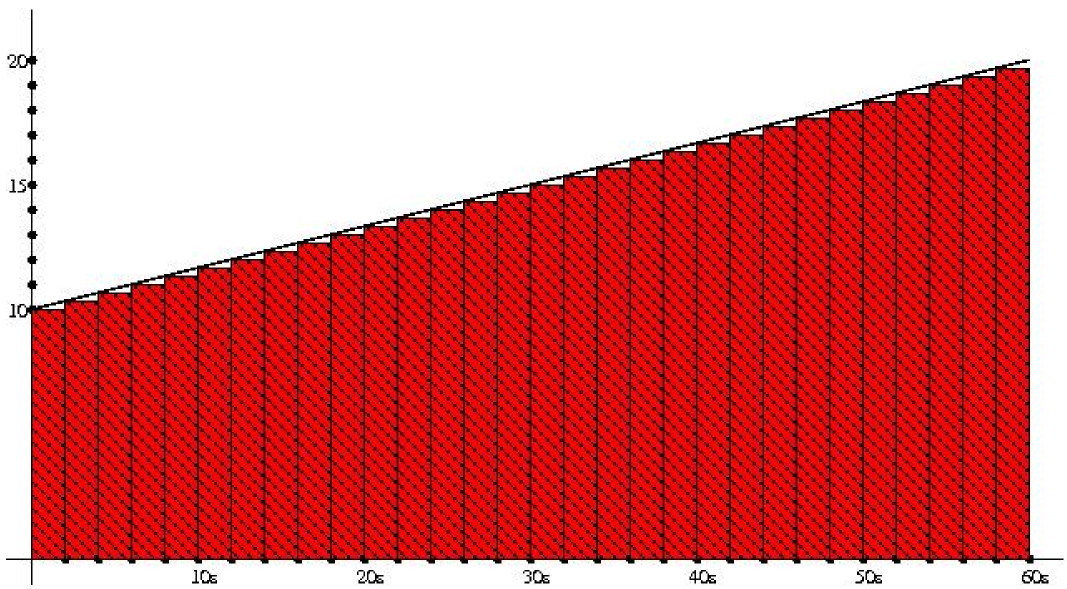
$$10 + \frac{10}{30} = 10\frac{1}{3}\mathrm{m/s}.$$

If we make the *simplifying assumption* that the speed remains constant at $10\frac{1}{3}\mathrm{m/s}$ throughout the second two-second interval, we estimate that the car travels $20\frac{2}{3}\mathrm{m}$ during the second two-second interval. This underestimates the true distance beacuse the car is actually accelerating *from* $10\frac{1}{3}\mathrm{m/s}$ during these two seconds.

- If we proceed in this manner we would estimate that the car travels
  - $20\mathrm{m}$ in the first two seconds;
  - $20\frac{2}{3}\mathrm{m}$ in the next two seconds;
  - $21\frac{1}{3}\mathrm{m}$ in the next two seconds, and so on;
  - $\ldots 39\frac{1}{3}\mathrm{m}$ in the 30th two-second interval.

This would give us a total of $890\mathrm{m}$ as the estimate for distance travelled, but that's not really the point of this discussion.

The distance that we estimate using the assumption that the speed remains constant for each of the 30 two-second intervals, is indicated by the area in red in the diagram below, where the black line is the true speed graph. Note that the red area includes all the area under the speed graph, except for 30 small triangles of base length 2 and height $\frac{1}{3}$.



Suppose now that we refine the estimate by dividing our minute of time into 60 one-second intervals and assuming the the speed remains constant for each of these, instead of into 30 two-second intervals.

If do this we will estimate that the car travels

- $10\mathrm{m}$ in the first seconds;
- $10\frac{1}{6}\mathrm{m}$ in the next seconds;
- $10\frac{2}{6}\mathrm{m}$ in the next second, and so on;
- $\ldots 19\frac{5}{6}\mathrm{m}$ in the 60th one-second interval.

This would give us a total of $895\mathrm{m}$ as the estimate for distance travelled. What is the corresponding picture? Draw it, or at least part of it, as an exercise.

Note that this *still underestimates* the distance travelled in each second, because it assumes that the speed remains constant at its starting point for the duration of each second, whereas in reality it increases. But this estimate is closer to the true answer than the last one, because this estimate takes into account speed increases every second, instead of every two seconds.

The corresponding "area" picture has sixty rectangles of width 1 instead of thirty of width 2, and it includes all the area under the speed graph, except for *sixty triangles of base length 1 and height* $\frac{1}{6}$.

If we used the same strategy but dividing our minute into 120 half-second intervals, we would expect to get a better estimate again. As the number of intervals increases and their width decreases, the red rectangles in the picture come closer and closer to filling *all* the area under the speed graph. The true distance travelled is the limit of these improving estimates, as the length of the subintervals approaches zero. This is *exactly* the area under the speed graph, between $x = 0$ and $x = 60$.

So we can now assert more confidently that the answer to Problem 1.1.4 is 900m.

Problem 1.1.5 *Again our car is travelling in one direction for one minute, but this time its speed* v *increases from* 10m/s *to* 20m/s *over the minute, according to the formula*

$$v(t) = 20 - \frac{1}{360}(60 - t)^2,$$

where t *is measured in seconds, and* $t = 0$ *at the start of the minute.*
What is the distance travelled?

NOTE: The formula means that after t seconds have passed, the speed of the car in m/s is $20 - \frac{1}{360}(60 - t)^2$. So for example after 30 seconds the car is travelling at a speed of

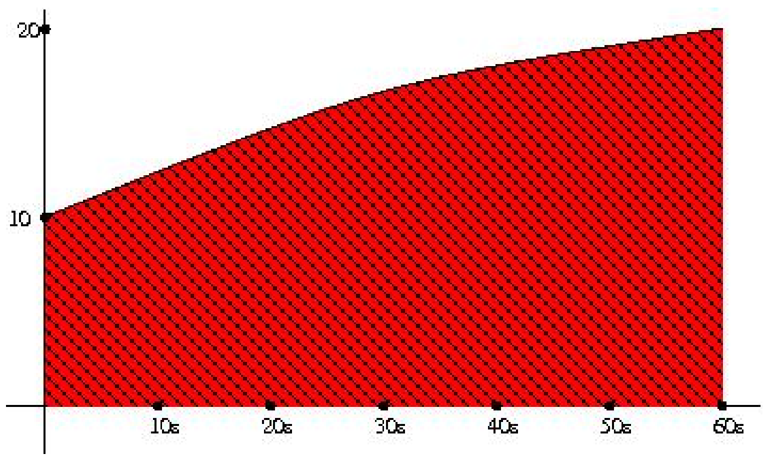
$$20 - \frac{1}{360}(60 - 30)^2 = 20 - \frac{1}{360}900 = 17.5\text{m/s}.$$

Note that $60 - t$ is decreasing as t increases from 0 to 60, so $(60 - t)^2$ is decreasing also. Thus the expression

$$20 - \frac{1}{360}(60 - t)^2$$

is *increasing* as t goes from 0 to 60. So the car is accelerating throughout the minute.

Below is the graph of the speed (in m/s) against time (in s), with the area below it (between $t = 0$ and $t = 60$) coloured red.



The argument above works in exactly the same way for this example, to persuade us that the distance travelled should be given by the area under the speed graph, between $t = 0$ and $t = 60$. This is the area that is coloured red in the picture above.

PROBLEM! The upper boundary of this area is a part of a parabola not a line segment. The region is not a combination of rectangles and triangles as in Problem 1.1.4. We can't calculate its area using elementary techniques.

So : what we need is a theory or a method that will allow us to calculate the area bounded by a section of the graph of a function and the x-axis, over a specified interval.

IMPORTANT NOTE: The problem of calculating the distance travelled by an object from knowledge of how its speed is changing is just one example of a scientific problem that can be solved by calculating the area of a region enclosed between a graph and the x-axis. Here are just a few more examples.

1. The fuel consumption of an aircraft is a function of its speed. The total amount of fuel consumed on a journey can be calculated as the area under the graph showing speed against time.

2. The energy stored by a solar panel is a function of the light intensity, which is itself a function of time. The total energy stored in one day can be modelled as the area under a graph of the light intensity against time for that day.

3. The volume of (for example) a square pyramid can be interpreted as the area of a graph of its horizontal cross-section area against height above the base.

4. In medicine, if a drug is administered intravenously, the quantity of the drug that is in the person's bloodstream can be calculated as the area under the graph of a function that depends both on the rate at which the drug is administered and on the rate at which it breaks down.

5. The quantity of a pollutant in a lake can be estimated by calculating areas under graphs of functions describing the rate at which the pollutant is being introduced and the the rate it which is is dispersing or being eliminated.

6. The concept of area under a graph is widely used in probability and statistics, where for example the probability that a randomly chosen person is aged between 20 and 30 years is the area under the graph of the appropriate *probability density function*, over the relevant interval.

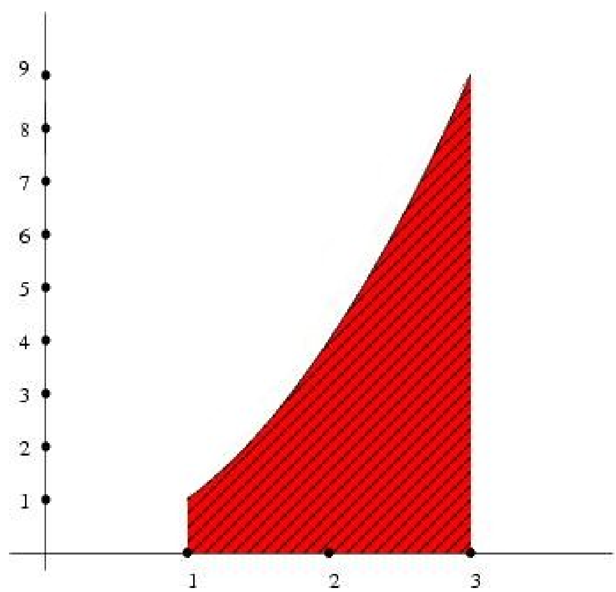
1.2 The Definite Integral

In the last section we concluded that a theory for discussing (and hopefully calculating) areas enclosed between the graphs of known functions and the x-axis, within specified intervals, would be useful. Such a theory does exist and it forms a large part of what is called integral calculus. In order to develop and use this theory we need a technical language and notation for talking about areas under curves. The goal of this section is to understand this notation and be able to use it - it is a bit cumbersome and not the most intuitively appealing, but with a bit of practice it is quite manageable.

Example 1.2.1 *Suppose that f is the function defined by $f(x) = x^2$. Note that $f(x)$ is positive when $1 \leqslant x \leqslant 3$. This means that in the region between the vertical lines $x = 1$ and $x = 5$, the graph $y = f(x)$ lies completely above the $x - \mathrm{axis}$. The area that is enclosed between the graph $y = f(x)$, the x-axis, and the vertical lines $x = 1$ and $x = 3$ is called the* definite integral of x^2 from $x = 1$ to $x = 3$, *and denoted by*

$$\boxed{\int_1^3 x^2\, dx.}$$

This diagram shows the region whose area is the definite integral $\int_1^3 x^2\, dx$.



NOTE: At the moment we are not trying to actually *calculate* this red area, we are just thinking about how the integral notation is used and what it means.

Example 1.2.2 *Suppose the function f is defined by*

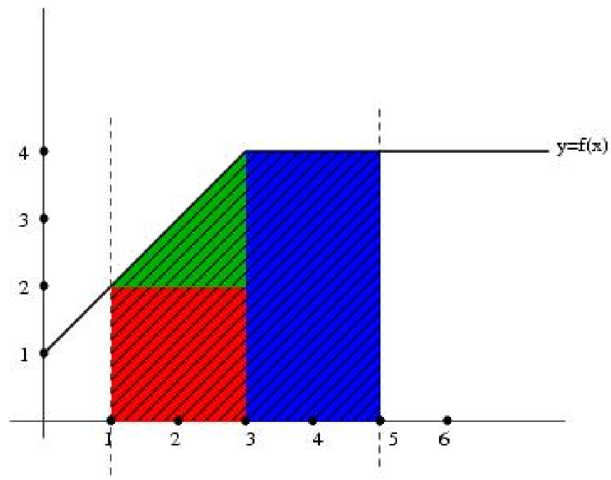
$$f(x) = \begin{cases} x + 1 & \text{if } 0 \leqslant x \leqslant 3 \\ 4 & \text{if } x \geqslant 3 \end{cases}$$

Then the graph of f consists of the section of the line $y = x+1$ between $x = 0$ and $x = 3$ (this is the line segment joining the points $(0, 1)$ and $(3, 4)$, and the constant line $y = 4$ from $x = 3$ onwards.

Now $\displaystyle\int_1^5 f(x)\, dx$ represents the area enclosed by the graph $y = f(x)$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. From the diagram below we can see that this area consists of

- A (green) triangle of base length 2 and height 2, area 2;

- A (red) rectangle of base length 2 and height 2, area 4;
- A (blue) rectangle of base length 2 and height 4, area 8.



Adding these three areas, we can conclude that

$$\int_1^5 f(x)\,dx = 2 + 4 + 8 = 14.$$

In this example we are able to calculate the actual value of the definite integral because the region whose area is involved is just an arrangement of rectangles and triangles. Note from this example that in general for a function f and numbers a and b, the definite integral $\int_a^b f(x)\,dx$ is *a number*.

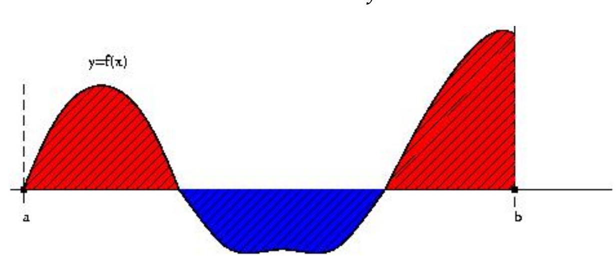
We now move on to the general definition of a definite integral.

Definition 1.2.3 *Let a and b be fixed real numbers, with $a < b$ (so a is to the left of b on the number line). Let f be a function for which it makes sense to talk about the* area *enclosed between the graph of f and the x-axis, over the interval from a to b. Then the* definite integral from a to b of f, *denoted*

$$\int_a^b f(x)\,dx$$

is defined to be the number obtained by subtracting the area enclosed below the x-axis by the graph $y = f(x)$ and the vertical lines $x = a$ and $x = b$ from the area enclosed above the x-axis by the graph $y = f(x)$ and the vertical lines $x = a$ and $x = b$.

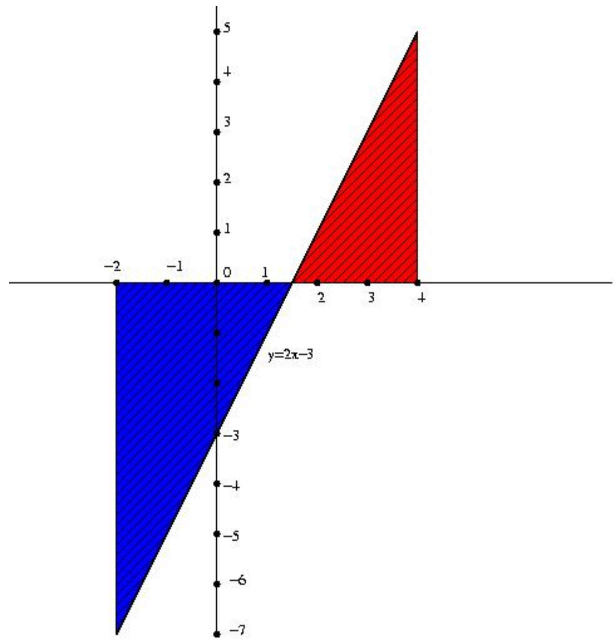
Example 1.2.4 *If the graph $y = f(x)$ is as shown in the diagram below, then $\displaystyle\int_a^b f(x)\,dx$ is the number obtained by subtracting the total area that is coloured blue from the total area that is coloured red.*



Example 1.2.5 *(a) Calculate $\displaystyle\int_{-2}^4 2x - 3\,dx$.*
(b) Calculate the total area enclosed between the x-axis and the line $y = 2x - 3$, between $x = -2$ and $x = 4$.

SOLUTION: (a) We need to describe the areas enclosed by the curve $y = 2x - 3$, above and below the x-axis, between $x = -2$ and $x = 4$.
The curve $y = 2x - 3$ is a line; it passes through the points $(-2, -7)$ and $(4, 5)$ and it intercepts the x-axis at $x = \frac{3}{2}$.

The diagram below describes the problem :



The area of the red triangle is

$$\frac{1}{2} \times \frac{5}{2} \times 5 = \frac{25}{4},$$

and the area of the blue triangle is

$$\frac{1}{2} \times \frac{7}{2} \times 7 = \frac{49}{4}.$$

Thus $\int_{-2}^{4} 2x - 3\,dx = \frac{25}{4} - \frac{49}{4} = -\frac{24}{4} = -6.$

(b) The *total area* enclosed between the x-axis and the line $y = 2x - 5$, between $x = -2$ and $x = 4$, is the sum of the areas of the red and blue triangles, which is $\frac{25}{4} + \frac{49}{4} = \frac{37}{2}$.

Note the difference between the two parts of this question, and be careful about this distinction.

NOTES

1. In Definition 1.2.3, What is meant by the phrase "for which it makes sense to talk about the *area* enclosed between the graph of f and the x-axis" is (more or less) that the graph $y = f(x)$ is not just a scattering of points, but consists of a curve or perhaps more than one curve. There is a formal theory about "integrable functions" that makes this notion precise.

2. *Note on Notation*
The notation surrounding definite integrals is a bit unusual. This note explains the various components involved in the expression

$$\int_{a}^{b} f(x)\,dx.$$

- "$\int$" is the *integral sign*.
- The "dx" indicates that f is a function of the variable x, and that we are talking about area between the graph of $f(x)$ against x and the x-axis.
- The "$f(x)$" in $\int_a^b f(x)\,dx$ is called the *integrand*. It is the function whose graph is the upper (or lower) boundary of the region whose area is being described.
- The numbers a and b are respectively called the *lower* and *upper* (or left and right) *limits of integration*. They determine the left and right boundaries of the region whose area is being described.

  In the expression $\int_a^b f(x)\,dx$, the limits of integration a and b are taken to be values of the variable x - this is included in what is to be interpreted from "dx". If there is any danger of ambiguity about this, you can write

  $$\int_{x=a}^{x=b} f(x)\,dx \text{ instead of } \int_a^b f(x)\,dx.$$

  Please do not confuse this use of the word "limit" with its other uses in calculus.

- Important note about this notation : neither the symbol "$\int$" nor the symbol "dx" in this setup is meaningful by itself : they must always accompany each other. You could think of them as being a bit like left and right parentheses or left and right quotation marks - a phrase that is opened with a left parenthesis "(" must be closed by a right parenthesis ")" - neither of these parentheses is meaningful by itself. In the language of definite integrals, an expression that is opened with the integral sign "$\int$" must be closed with the symbol "dx" (or "dt" or "du" as appropriate) indicating the variable involved. The symbol "dx" doesn't really have a meaning by itself - it is a companion to the integral sign.

Some Historical Remarks

The notation that is currently in use for the definite integral was introduced by Gottfried Leibniz around 1675. The rationale for it is as follows :
Areas were estimated as we did in Section 1.1. The interval from a to b would be divided into narrow subintervals, each of width Δx. The name x_i would be given to the left endpoint of the ith subinterval, and the height of the graph above the point x_i would be given by $f(x_i)$. So the area under the graph on this ith subinterval would be approximated by that of a rectangle of width Δx and height $f(x_i)$. The total area would be approximated by the sum of the areas of all of these narrow rectangles, which was written as

$$\sum f(x_i)\Delta x.$$

The accuracy of this estimate improves as the width of the subintervals gets smaller and the number of them gets larger; the true area is the limit of this process as $\Delta x \to 0$. The notation "dx" was introduced as an expression to replace Δx in this limit, and the integral sign $\int$ is a "limit version" of the summation sign $\sum$. The integral symbol itself is based on the "long s" character which was in use in English typography until about 1800.

The idea of calculating areas of regions by taking finer and finer subdivisions in this manner dates back to the ancient Greeks; early examples of what is now called "integration" can be found in the work of Archimedes (circa 225 BC). The idea of computing areas under graphs by taking narrower and narrower vertical columns was put on a firm theoretical basis by Bernhard Riemann in the 1850s.

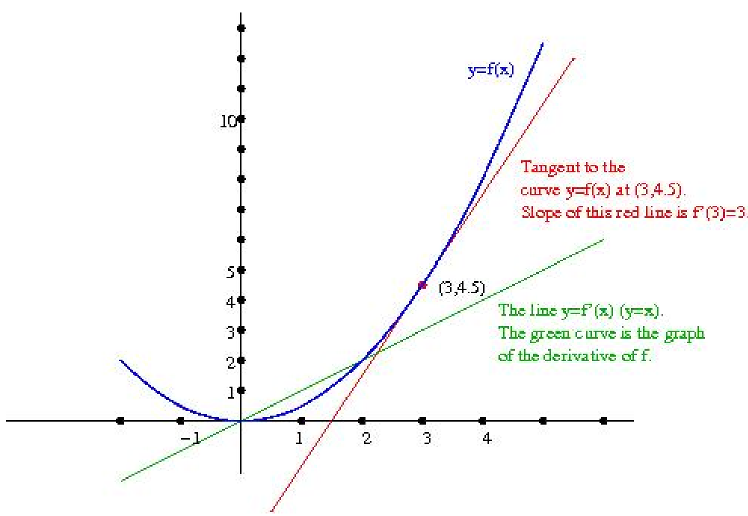
For more information on the history of calculus and of mathematics generally, see `http://www-history.mcs.st-and.ac.uk/index.html`.

1.3 The Fundamental Theorem of Calculus

In this section, we discuss the *Fundamental Theorem of Calculus* which establishes a crucial link between differential calculus and the problem of calculating definite integrals, or areas under curves. At the end of this section, you should be able to explain this connection and demonstrate with some examples how the techniques of differential calculus can be used to calculate definite integrals.

Differential calculus is about *how functions are changing*. Suppose for example, that you are thinking of temperature (in °C) as a function of time (in hours). You might write temperature as $T(t)$ to indicate that the temperature T varies with time t. The *derivative* of the function $T(t)$, denoted $T'(t)$, tells us how the temperature is *changing over time*. If you know that at 10.00am yesterday the derivative of T was 0.5 (°C/hr), then you know that the temperature was *increasing* by half a degree per hour at that time. However this does not tell you anything about what the temperature actually was at this time. If you know that by 10.00pm last night the derivative of the temperature was -2°C/hr you still don't know anything about what the temperature was at the time, but you know that it was cooling at a rate of 2 degrees per hour. The derivative T' is itself a function of time, as the rate of increase or decrease of temperature will not remain constant throughout the day. Knowing about $T'(t)$ doesn't tell us anything about how warm or cold it was at any given time, but it gives us such information as when it was getting warmer, when it was getting colder, when it stopped getting warmer and started to cool, and so on.

RECALL : Suppose that f is a function of a variable x. Then $f'(x)$ is the *derivative* of f, also a function of x. The value of f' at a particular point a is the *slope* of the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$. The diagram below shows the graph of the function defined by $f(x) = \frac{1}{2}x^2$ and the tangent line to this graph at the point (3,4.5). The *slope* of this tangent line (which happens to be 3) is the *derivative* of f when $x = 3$, i.e. it is $f'(3)$. As x varies – as we move along the graph from left to right – the slope of the tangent line varies too, so f' is a function of x; as we know it is given in this example by the formula $f'(x) = \dfrac{d}{dx}\left(\dfrac{1}{2}x^2\right) = x$.



Now we are going to define a new function related to definite integrals and consider its derivative - we start with an example. Recall Problem 7 from the "Practice Problems 1" sheet.

Example 1.3.1 *At time* $t = 0$ *an object is travelling at 5 metres per second. After* t *seconds its speed in* m/s *is given by*

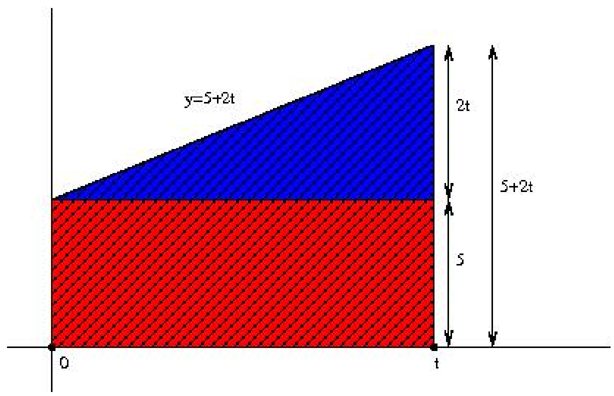
$$v(t) = 5 + 2t.$$

Let $s(t)$ denote the distance travelled by the object after t seconds. So $s(t)$ depends on t obviously since the object is moving over time. From our work in Section 1.1 we know that $s(t)$ is the area under the graph of $v(t)$ against t, between the vertical lines through 0 and t. We can calculate this in terms of t, by drawing a picture of the graph.

Look at the shape of the region between the graph and the x-axis, between the vertical lines through 0 and t. It is a trapezoid with

- bottom edge formed by a segment of the x-axis of length t;
- left and right edges formed by segments of the vertical lines through 0 and t, of lengths 5 and $5 + 2t$ respectively;
- Top edge formed by part of the graph $y = 5 + 2t$.

The area of this region is $s(t)$. As shown in the diagram, it is the sum of the areas of a rectangle of width t and height 5 (area 5t) and a triangle of width t and height 2t (area t^2). This means : *for any $t \geqslant 0$, the distance covered by this object in the first t seconds of its movement is given by $s(t) = 5+t^2$.*



IMPORTANT NOTE: The function $s(t)$ associates to t the area under the graph $y = v(t)$ from time 0 to time t. As t increases (i.e. as time passes), this area increases (it represents the distance travelled which is obviously increasing). Note that the derivative of $s(t)$ is exactly $v(t)$.

$$s(t) = 5t + t^2;\ s'(t) = 5 + 2t = v(t).$$

We shouldn't really be surprised by this given the physical context of the problem : $s(t)$ is the total distance travelled at time t, and $s'(t)$ at time t is $v(t)$, the speed at time t. So this is saying that the *instantaneous rate of change* of the distance travelled at a particular moment is the *speed* at which the object is travelling at that moment - which makes sense.

However, there is another way to interpret this statement, which makes sense for definite integrals generally :

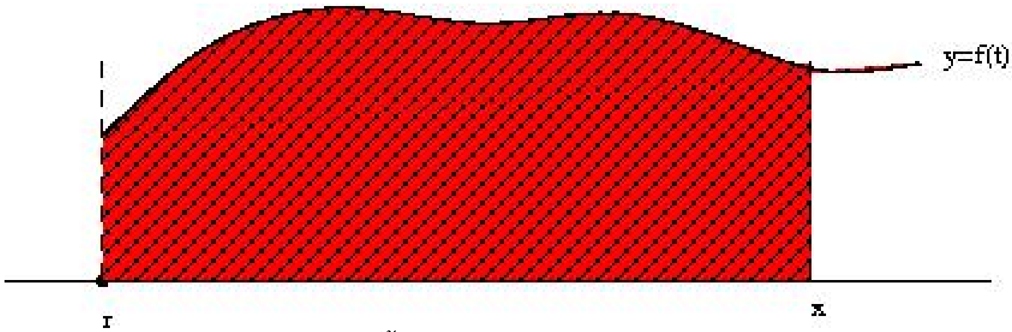
- v is a function whose graph we are looking at.
- For a positive number t, $s(t)$ is the area under the graph of v, to the right of 0 and to the left of t.
- Then the derivative of s is just v, the function under whose graph the area is being measured, i.e $s'(t) = v(t)$.

The more general version of this statement is the *Fundamental Theorem of Calculus*, stated below.

Theorem 1.3.2 *(Fundamental Theorem of Calculus (FToC))*
Let f be a (suitable) function, and let r be a fixed number. Define a function A by

$$A(x) = \int_r^x f(t)\,dt.$$

This means : for a number x, $A(x)$ is the area enclosed by the graph of f and the $x - \mathrm{axis}$, between the vertical lines through r and x. The picture below shows what the function A does.



$A(x) = \int_r^x f(t)\,dt$ *is the area shown in red.*

The function A *depends on the variable* x*, via the right limit in the definite integral. The Fundamental Theorem of Calculus tells us that the function* s *is exactly the derivative of this area accumulation function* A*. Thus*

$$A'(x) = f(x).$$

NOTES

1. We won't prove the Fundamental Theorem of Calculus, but to get a feeling for what it says, look again at the picture above, and think about how $A(x)$ changes when x moves a little to the right. If $f(x) = 0$, $A(x)$ doesn't change at all as no area is accumulating under the graph of f. If $f(x)$ is positive and large, $A(x)$ increasees quickly as x moves to the right. If $f(x)$ is positive but smaller, $A(x)$ increases more slowly with x, because area accumulates more slowly under the "lower" curve. If $f(x)$ is negative, then $A(x)$ will decrease as x increases, because we will be accumulating "negative" area.

2. The Fundamental Theorem of Calculus is *interesting* because it connects differential calculus to the problem of calculating definite integrals, or areas under curves.

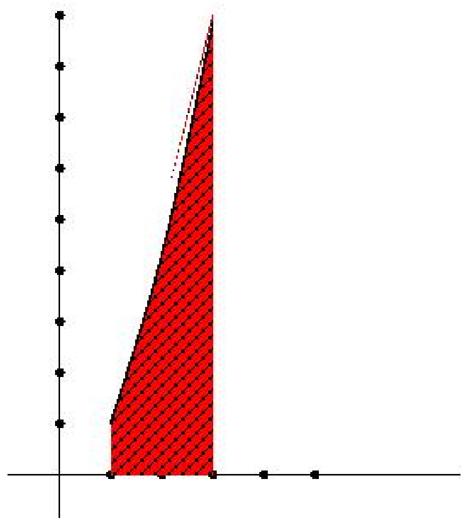
3. The Fundamental Theorem of Calculus is *useful* because we know a lot about differential calculus. Using the machinery of differentiation (the product rule, chain rule etc), we can calculate the derivative of just about anything that can be written in terms of elementary functions (like polynomials, trigonometric functions, exponentials and so on). So we have a lot of theory about differentiation that is all of a sudden relevant to calculating definite integrals as well.

4. The Fundamental Theorem of Calculus can be traced back to work of *Isaac Barrow* and *Isaac Newton* in the mid 17th Century.

Finally we show how to use the Fundamental Theorem of Calculus to calculate definite integrals.

Example 1.3.3 *Calculate* $\int_1^3 x^2\,dx$*.*

Solution: The area that we want to calculate is shown in the picture below.



Imagine that r is some point to the left of 1, and that the function A is defined for $x \geqslant r$ by

$$A(x) = \int_r^x x^2\,dx,$$

i.e. $A(x)$ is the area under the graph of x^2 between r and x. Then

$$\int_1^3 x^2\,dx = A(3) - A(1);$$

this is the area under the graph that is to the left of 3 but to the right of 1. So - if we could calculate $A(x)$, we could evaluate this function at $x = 3$ and at $x = 0$.

What we know about the function $A(x)$, from the Fundamental Theorem of Calculus, is that its derivative is given by $A'(x) = x^2$. What function A has derivative x^2?

The derivative of x^3 is $3x^2$, so the derivative of $\frac{1}{3}x^3$ is x^2.

Note : $\frac{1}{3}x^3$ is *not the only* expression whose derivative is x^2. For example $\frac{1}{3}x^3 + 1$, $\frac{1}{3}x^3 - 5$ and any expression of the form $\frac{1}{3}x^3 + C$ for any constant C, also have derivative x^2. All of these are candidates for $A(x)$: basically they just correspond to different choices for the point r. All of these choices for $A(x)$ give the same outcome when we use them to evaluate $\int_1^3 x^2\,dx$ as suggested above.

So : take $A(x) = \frac{1}{3}x^3$. Then

$$\int_1^3 x^2\,dx = A(3) - A(1) = \frac{1}{3}(3^3) - \frac{1}{3}(1^3) = 9 - \frac{1}{3} = \frac{26}{3}.$$

This technique is described in general terms in the following version of the Fundamental Theorem of Calculus :

Theorem 1.3.4 *(Fundamental Theorem of Calculus, Part 2) Let f be a function. To calculate the definite integral*

$$\int_a^b f(x)\,dx,$$

first find a function F whose derivative is f, i.e. for which $F'(x) = f(x)$. (This might be hard). Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Chapter 2

Techniques of Integration

2.1 Getting Started - Uncomplicated Examples

Recall the following strategy for evaluating definite integrals, which arose from the Fundamental Theorem of Calculus (see Section 1.3). To calculate

$$\int_a^b f(x), dx$$

1. Find a function F for which $F'(x) = f(x)$, i.e. find a function F whose derivative is f.
2. Evaluate F at the limits of integration a and b; i.e. calcuate $F(a)$ and $F(b)$. This means replacing x separately with a and b in the formula that defines $F(x)$.
3. Calculate the number $F(b) - F(a)$. This is the definite integral $\int_a^b f(x)\,dx$.

Of the three steps above, the first one is the hard one. There are many examples of (very reasonable looking) functions f for which it is not possible to write down a function F whose derivative is f in a manageable way. But there are many also for which it is, and they will be the focus of our attention in this chapter.

Suppose for example we look at the function g defined by $g(x) = \sin(x^2 + x)$. From the chain rule for differentiation we know that $g'(x) = (2x + 1)\cos(x^2 + x)$. But suppose that we started with

$$(2x + 1)\cos(x^2 + x)$$

and we wanted to find something whose derivative with respect to x was equal to this expression. *How would we get back to* $\sin(x^2 + x)$? In this chapter we will develop answers to this question, but it doesn't have a neat answer. The answer consists of a collection of strategies, techniques and observations that have to be employed judiciously and adapted for each example. It takes some careful practice to become adept at reversing the differentiation process which is basically what we have to do.

Recall the following notation from the supplementary problems to Chapter 1 : if F is a function that satisfies $F'(x) = f(x)$, then

$$f(x)|_a^b \text{ or } f(x)|_{x=a}^{x=b} \text{ means } F(b) - F(a).$$

We also need the following definitions :

Definition 2.1.1 *Let* f *be a function. Another function* F *is called an* antiderivative *of* f *if the derivative of* F *is* f*, i.e. if* $F'(x) = f(x)$*, for all (relevant) values of the variable* x*.*

Thus for example x^2 is an antiderivative of $2x$. Note that $x^2 + 1$, $x^2 + 5$ and $x^2 - 20e$ are also antiderivatives of $2x$. So we talk about *an* antiderviative of a function or expression rather that *the* antiderivative. So : a function may have more than one antiderivative, but different antiderivatives of a particular function will always differ from each other by a constant.
Note : Two functions will have the same derivative if their graphs differ from each other only by a vertical shift; in this case the tangent lines to these graphs for particular values of x will always have the same slope.

Definition 2.1.2 *Let* f *be a function. The* indefinite integral *of* f, *written*

$$\int f(x)\,dx$$

is the "general antiderivative" of f. *If* $F(x)$ *is a particular antiderivative of* f, *then we would write*

$$\int f(x)\,dx = F(x) + C,$$

to indicate that the diffent antiderivatives of f *look like* $F(x) + C$, *where* C *maybe any constant. (In this context* C *is often referred to as a constant of integration).*

Example 2.1.3 *We would write*

$$\int 2x\,dx = x^2 + C$$

to indicate that every antiderivative of $2x$ *has the form* x^2+C *for some constant* C, *and that every expression of the form* $x^2 + C$ *(for a constant* C*) has derivative equal to* $2x$.

In this section we will consider examples where antiderivatives can be determined without recourse to any sophisticated techniques (which doesn't necessarily mean easily).

The following table reminding us of the derivatives of some elementary functions may be helpful.

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
$\frac{1}{x^2}$	$-\frac{2}{x^3}$
x^n	nx^{n-1}

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\sin 2x$	$2\cos 2x$
e^x	e^x
e^{3x}	$3e^{3x}$

Basically our goal is to figure out how to get from the right to the left column in a table like this.

Example 2.1.4 *Find (i)* $\int x^2\,dx$, *(ii)* $\int_4^6 x^2\,dx$

SOLUTION: (i) $\frac{d}{dx}(x^3) = 3x^2$ - so x^3 is not an antiderivative of x^2, it is "too big" by a factor of 3. Thus $\frac{1}{3}x^3$ should be an antiderivative of x^2; indeed

$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = \frac{1}{3}3x^2 = x^2.$$

We conclude

$$\int x^2\,dx = \frac{1}{3}x^3 + C.$$

This means that every antiderivative of x^2 has the form $\frac{1}{3}x^3 + C$ for some constant C.

(ii) By FTC (Part 2) we have

$$\int_4^6 x^2\,dx = \left.\frac{x^3}{3}\right|_4^6 = \frac{6^3}{3} - \frac{4^3}{3} = \frac{153}{3}.$$

Example 2.1.5 *Determine* $\int \cos 2x\,dx$.

SOLUTION: The question is : what do we need to differentiate to get $\cos 2x$? Well, what do we need to differentiate to get something involving cos?
(If you can't answer this question fairly quickly, you are advised to brush up on your knowledge of derivatives of trigonometric functions - don't forget that the SUMS centre can help in this situation).
We know that the derivative of $\sin x$ is $\cos x$.
So a reasonable guess would say that the derivative of $\sin 2x$ might be "something like" $\cos 2x$.
By the chain rule, the derivative of $\sin 2x$ is in fact $2\cos 2x$.
So, in our search for an antiderivative of $\cos 2x$, $\sin 2x$ is pretty close but it gives us twice what we want - we are out by a factor of 2.

So we should compensate for this by taking $\frac{1}{2}\sin 2x$; its derivative is

$$\frac{1}{2}(2\cos 2x) = \cos 2x.$$

CONCLUSION: $\int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$.
NOTE: The reason for the commentary on this example is to give you an idea of the sorts of thought processes a person might go through while figuring out an antiderivative of $\cos 2x$. You would not be expected to provide this sort of commentary if you were answering a question like this in an assessment - it would be enough to just write the line labelled "CONCLUSION" above.
The following examples are similar, with less commentary provided as we continue.

Example 2.1.6 *Determine* $\int e^{\frac{1}{2}x}\,dx$

SOLUTION: We are looking for something whose derivative is $e^{\frac{1}{2}x}$. We know that the derivative of e^x is e^x, so the answer should be something like $e^{\frac{1}{2}x}$. But this is not exactly right because the derivative of $e^{\frac{1}{2}x}$ is

$$\frac{1}{2}e^{\frac{1}{2}x},$$

which is only half of what we want - we are out by a factor of $\frac{1}{2}$ - what we want is twice what we have. We can compensate for this by multiplying what we have by 2 (or dividing it by $\frac{1}{2}$ which is the same). So what we want is $2e^{\frac{1}{2}x}$ - use the chain rule to confirm that the derivative of this expression is $e^{\frac{1}{2}x}$ as required.

CONCLUSION: $\int e^{\frac{1}{2}x}\,dx = 2e^{\frac{1}{2}x} + C$

Example 2.1.7 *Determine* $\int x^5\,dx$

SOLUTION: The derivative of x^6 is $6x^5$. So the derivative of $\dfrac{1}{6}x^6$ is x^5. Hence

$$\int x^5\,dx = \frac{1}{6}x^6 + C.$$

IMPORTANT NOTE: We know that in order to calculate the derivative of an expression like x^n, we reduce the index by 1 to $n-1$, and we multiply by the constant n. So

$$\frac{d}{dx}x^n = nx^{n-1}$$

in general. To find an *antiderivative* of x^n we have to reverse this process. This means that the index *increases* by 1 to $n+1$ and we multiply by the constant $\mathtt{disfrac}1n+1$. So

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$

This makes sense as long as the number n is not equal to -1 (in which case the fraction $\frac{1}{n+1}$ wouldn't be defined). We will see later how to manage $\int x^{-1}\,dx$ or $\int \frac{1}{x}\,dx$.
Note: included in the general description of $\int x^n\,dx$ above is the statement that

$$\int 1\,dx = x + C.$$

This makes sense when we ask ourselves what we need to differentiate in order to get 1. The answer is x.

Example 2.1.8 *Determine* $\int 3x^2 + 2x + 4\,dx.$

SOLUTION: $\int 3x^2 + 2x + 4\,dx = 3(x^3/3) + 2(x^2/2) + 4x + C = x^3 + 2x^2 + 4x + C.$
Remark: Here we are separately applying our ability to integrate expressions of the form x^n to the x^3 term, the x^2 term, and the constant term. We are also making use of the following fact that indefinite integration behaves *linearly*. This means : if $f(x)$ and $g(x)$ are expressions involving x and a and b are real numbers, we have

$$\int af(x) + bg(x)\,dx = a\int f(x)\,dx + b\int g(x)\,dx.$$

Example 2.1.9 *Determine* $\int_0^{\pi} \sin x + \cos x\,dx.$

SOLUTION: We need to write down *any* antiderivative of $\sin x + \cos x$ and evaluate it at the limits of integration :

$$\begin{aligned}\int_0^{\pi} \sin x + \cos x\,dx &= -\cos x + \sin x|_0^{\pi} \\ &= (-\cos\pi + \sin\pi) - (-\cos 0 + \sin 0) \\ &= -(-1) + 0 - (-1 + 0) = 2.\end{aligned}$$

NOTE: In case you struggle to remember things like cosine and sine of π, $\frac{\pi}{2}$ etc, it is easy enough if you think about it in terms of the definitions of the trigonometric functions. To determine $\cos\pi$, start at the point $(1, 0)$ and travel counter–clockwise around the unit circle through an angle of π radians (180 degrees), arriving at the point $(-1, 0)$. The x-coordinate of the point you are at now is $\cos\pi$, and the y-coordinate is $\sin\pi$.

Example 2.1.10 *Determine* $\int x^{1/3}\,dx.$

SOLUTION: $\int x^{1/3}\,dx = \dfrac{1}{4/3}x^{4/3} + C = \dfrac{3}{4}x^{4/3} + C.$

2.2 Substitution - Reversing the Chain Rule

The Chain Rule of Differentation tells us that in order to differentiate the expression $\sin x^2$, we should regard this expression as sin("something") whose derivative (with respect to "something" is cos("something"), then multiply this expression by the derivative of the "something" with respect to x. Thus

$$\frac{d}{dx}(\sin x^2) = \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2.$$

Equivalently

$$\int 2x \cos x^2 \, dx = \sin x^2 + C.$$

In this section, through a series of examples, we consider how one might go about reversing the differentiation process to get from $2x \cos x^2$ back to $\sin x^2$.

Example 2.2.1 *Determine* $\int 2x\sqrt{x^2+1}\, dx$.

SOLUTION: Notice that the integrand (i.e. the expression to be integrated) involves both the expressions x^2+1 and $2x$. Note also that $2x$ is the derivative of x^2+1.
Introduce the notation u and set $u = x^2+1$. Note $\dfrac{du}{dx} = 2x$.
Then $2x\sqrt{x^2+1} = \frac{du}{dx}\sqrt{u} = u^{\frac{1}{2}}\dfrac{du}{dx}$.

Suppose we were able to find a function F of u for which $\dfrac{d}{du}(F(u)) = u^{\frac{1}{2}}$. Then by the chain rule we would have

$$\frac{d}{dx}(F(u)) = \frac{d}{du}(F(u))\frac{du}{dx} = u^{\frac{1}{2}}(2x) = 2x\sqrt{x^2+1}.$$

So $F(u)$ would be an antiderivative (with respect to x) of $2x\sqrt{x^2+1}$.
Thus we want

$$\frac{d}{du}(F(u)) = u^{\frac{12}{.}}$$

So take

$$F(u) = \int u^{\frac{1}{2}}\, du = \frac{2}{3}u^{\frac{3}{2}} + C.$$

(At this stage we are just using the note after Example 2.1.7, with $n = \frac{1}{2}$).
Thus

$$\int 2x\sqrt{x^2+1}\, dx = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C.$$

We usually formulate this procedure of "integration by substitution" in the following more concise way.
To find $\int 2x\sqrt{x^2+1}\, dx.$:
Let $u = x^2+1$.
Then $\dfrac{du}{dx} = 2x \Longrightarrow du = 2x\, dx$. Then

$$\int 2x\sqrt{x^2+1}\, dx = \int \sqrt{x^2+1}(2x\, dx) = \int u^{\frac{1}{2}}\, du = \frac{2}{3}u^{\frac{3}{2}} + C.$$

So

$$\int 2x\sqrt{x^2+1}\, dx = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C.$$

Example 2.2.2 *Determine* $\int x\sin(2x^2)\,dx$

SOLUTION: Let $u = 2x^2$.
Then $\frac{du}{dx} = 4x\,dx;\ x\,dx = \frac{1}{4}du$. So

$$\int x\sin(2x^2)\,dx = \frac{1}{4}\int \sin u\,du = -\frac{1}{4}\cos u + C = \frac{1}{4}\cos(2x^2) + C.$$

REMARK: It is good practice to check your answer to a problem like this, either mentally or on paper. Check that the derivative of $-\frac{1}{4}\cos(2x^2)$ is indeed equal to $x\sin(2x^2)$.

Example 2.2.3 *Determine* $\int (1-\cos t)^2 \sin t\,dt$

SOLUTION: Write $u = 1-\cos t$.
Then $\frac{du}{dt} = sint;\ du = \sin t\,dt$.
So

$$\int (1-\cos t)^2\sin t\,dt = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(1-\cos t)^3 + C.$$

QUESTION: How do we know what expression to extract and refer to as u?
Really what we are doing in this process is changing the integration problem in the variable t to a (hopefully easier) integration problem in a new variable u - there is a change of variables taking place.

There is no easy answer to the question of how to decide what to rename as "u", but with practice we can develop a sense of what might work. In this example the integrand involves the expression $1-\cos t$ and also its derivative $\sin t$. This is what makes the substitution $u = 1-\cos t$ effective for this problem. The "$\sin t$" part of the integrand gets "absorbed" into the "du" in the change of variables, and the "$1-\cos t$" part is obviously easily written in terms of u. We could try the alternative $u = \sin t$, but this is not likely to be helpful, since it is not so easy to see how to express $1-\cos t$ in terms of this u, or what would happen with du which would be effectively $\cos t\,dt$.

Example 2.2.4 *To determine* $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx$

SOLUTION: How are we to choose u? Well, what are the candidates?
The integrand involves the expressions $1+\sqrt{x}$ and $\frac{1}{\sqrt{x}}$. The derivative of $1+\sqrt{x}$ is "something like" $\frac{1}{\sqrt{x}}$, so setting $u = 1+\sqrt{x}$ might be worth a try.

Let $u = 1+\sqrt{x}$.
Then $\frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}};\ \frac{1}{\sqrt{x}}dx = 2\,du$.
So

$$\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx = 2\int u^3\,du = \frac{2}{4}u^4 + C = \frac{1}{2}(1+\sqrt{x})^4 + C.$$

Example 2.2.5 *Determine* $\int \frac{16x}{\sqrt{8x^2+1}}\,dx$

SOLUTION: Let $u = \sqrt{8x^2+1}$.
Then $\frac{du}{dx} = \frac{1}{2}(8x^2+1)^{-\frac{1}{2}}(16x) = \frac{8x}{\sqrt{8x^2+1}}$.
Thus $\frac{16x}{\sqrt{8x^2+1}}\,dx = 2du$, and

$$\int \frac{16x}{\sqrt{8x^2+1}}\,dx = 2\int du = 2u + C = 2\sqrt{x^2+1} + C.$$

NOTE: An alternative here would have been to set $u = 8x^2+1$. That this would also be successful is left for you to check as an exercise.

DIGRESSION - IMPORTANT NOTE: *The Integral* $\int \frac{1}{x}\,dx$

Suppose that $x > 0$ and $y = \ln x$. Recall this means (by definition) that $e^y = x$. Differentiating both sides of this equation (with respect to x) gives

$$e^y \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Thus the derivative of $\ln x$ is $\frac{1}{x}$, and

$$\int \frac{1}{x}\,dx = \ln x + C, \text{ for } x > 0.$$

If $x < 0$, then

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

This latter formula applies for all $x \neq 0$.

Example 2.2.6 *To determine* $\int \frac{\sec^2 x}{\tan x}\,dx$

Note : the derivative of $\tan x$ is $\sec^2 x$, suggesting the substitution $u = \tan x$. You are not necessarily expected to know the derivative of $\tan x$ (or of any of the trigonometric functions) of the top of your head, but you should know where to find them in the "Formulae and Tables" booklet.

Let $u = \tan x$.

Then $\frac{du}{dx} = \sec^2 x;\ du = \sec^2 x\,dx$. Thus $\frac{\sec^2 x}{\tan x}\,dx = \frac{1}{u}du$, and

$$\int \frac{\sec^2 x}{\tan x}\,dx = \int \frac{1}{u}\,du = \log|u| + C = \log|\tan x| + C.$$

SUBSTITUTION AND DEFINITE INTEGRALS

Example 2.2.7 *Evaluate* $\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$.

SOLUTION: To find an antiderivative, let $u = 4 + r^2$.

Then $\frac{du}{dr} = 2r,\ du = 2r\,dr;\ 5r\,dr = \frac{5}{2}du$.

So

$$\int \frac{5r}{(4+r^2)^2}\,dr = \frac{5}{2}\int \frac{1}{u^2}\,du = \frac{5}{2}\int u^{-2}\,du.$$

Thus $\int \frac{5r}{(4+r^2)^2}\,dr = -\frac{5}{2} \times \frac{1}{u} + C$, and we need to evaluate $-\frac{5}{2} \times \frac{1}{u}$ at $r = 0$ and at $r = 1$. We have two choices :

1. Write $u = 4 + r^2$ to obtain

$$\begin{aligned}
\int_0^1 \frac{5r}{(4+r^2)^2}\,dr &= -\frac{5}{2}\frac{1}{4+r^2}\bigg|_{r=0}^{r=1} \\
&= -\frac{5}{2}\frac{1}{4+1^2} - \left(-\frac{5}{2} \times \frac{1}{4+0^2}\right) \\
&= -\frac{5}{2} \times \frac{1}{5} + \frac{5}{2} \times \frac{1}{4} \\
&= \frac{1}{8}.
\end{aligned}$$

2. Alternatively, write the antiderivative as $-\frac{5}{2}\frac{1}{u}$ and replace the limits of integration with the corresponding values of u.
When $r = 0$ we have $u = 4 + 0^2 = 4$.
When $r = 1$ we have $u = 4 + 1^2 = 5$.
Thus

$$\begin{aligned}
\int_0^1 \frac{5r}{(4+r^2)^2}\,dr &= -\frac{5}{2}\times\frac{1}{u}\bigg|_{u=4}^{u=5} \\
&= -\frac{5}{2}\times\frac{1}{5} - \left(-\frac{5}{2}\times\frac{1}{4}\right) \\
&= \frac{1}{8}.
\end{aligned}$$

2.3 Partial Fraction Expansions - Integrating Rational Functions

We know how to integrate polynomial functions; for example

$$\int 2x^2 + 3x - 4\,dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 4x + C.$$

We also know that

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

This section is about integrating *rational functions*; i.e. quotients in which the numerator and denominator are both polynomials.

REMARK: If we were presented with the task of adding the expressions $\frac{2}{x+3}$ and $\frac{1}{x+4}$, we would take $(x+3)(x+4)$ as a *common denominator* and write

$$\frac{2}{x+3} + \frac{1}{x+4} = \frac{2(x+4)}{(x+3)(x+4)} + \frac{1(x+3)}{(x+3)(x+4)} = \frac{2(x+4)+1(x+3)}{(x+3)(x+4)} = \frac{3x+11}{(x+3)(x+4)}.$$

Question: Suppose we were presented with the expression $\frac{3x+11}{(x+3)(x+4)}$ and asked to rewrite it in the form $\frac{A}{x+3} + \frac{B}{x+4}$, for *numbers* A and B. How would we do it?

Another Question Why would we want to do such a thing?

Answer to the second question: Maybe if we want to integrate the expression : we know how to integrate things like $\frac{1}{x+3}$, but not things like $\frac{3x+11}{(x+3)(x+4)}$.

Answer to the first question: Write

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}.$$

Then

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A(x+4)}{(x+3)(x+4)} + \frac{B(x+3)}{(x+3)(x+4)} = \frac{(A+B)x + 4A + 3B}{(x+3)(x+4)}.$$

This means $3x + 11 = (A+B)x + 4A + 3B$ for all x, which means

$$A + B = 3, \text{ and } 4A + 3B = 11.$$

Thus $-4A - 4B = -12,\ -B = -1,\ B = 1$ and $A = 2$. So

$$\frac{3x+11}{(x+3)(x+4)} = \frac{2}{x+3} + \frac{1}{x+4}.$$

Alternative Method: We want

$$3x + 11 = A(x+4) + B(x+3),$$

for *all* real numbers x. If this statement is true for all x, then in particular it is true when $x = -4$. Setting $x = -4$ gives

$$-12 + 11 = A(0) + B(-1) \Longrightarrow B = 1.$$

Setting $x = -3$ gives

$$-9 + 11 = A(1) + B(0) \Longrightarrow A = 2.$$

Thus

$$\frac{3x+11}{(x+3)(x+4)} = \frac{2}{x+3} + \frac{1}{x+4}.$$

Expansions of rational functions of this sort are called *partial fraction expansions*.

Example 2.3.1 *Determine* $\displaystyle\int \frac{3x+11}{(x+3)(x+4)}\,dx$.

SOLUTION : Write
$$\int \frac{3x+11}{(x+3)(x+4)}\,dx = \int \frac{2}{x+3}\,dx + \int \frac{1}{x+4}\,dx$$
Then
$$\int \frac{3x+11}{(x+3)(x+4)}\,dx = 2\ln|x+3| + \ln|x+4| + C = \ln(x+3)^2 + \ln|x+4| + C.$$

Example 2.3.2 *Determine* $\displaystyle\int \frac{1}{x^2+5x+6}\,dx$.

SOLUTION: Write $\displaystyle\frac{1}{x^2+5x+6} = \frac{1}{(x+2)(x+3)}$ in the form
$$\frac{A}{x+2} + \frac{B}{x+3},$$
for constants A and B. This means
$$\frac{1}{(x+2)(x+3)} = \frac{A(x+3)+B(x+2)}{(x+2)(x+3)},$$
i.e. $1 = A(x+3) + B(x+2)$ for all x.
Thus
$$0x + 1 = (A+B)x + (3A+2B),$$
which means $A+B=0$ and $3A+2B=1$. This pair of equations has the unique solution $A = 1,\ B = -1$. Thus
$$\begin{aligned}
\frac{1}{(x+2)(x+3)} &= \frac{1}{x+2} = \frac{1}{x+3} \\
\Longrightarrow \int \frac{1}{(x+2)(x+3)} &= \int \frac{1}{x+2} - \frac{1}{x+3}\,dx \\
&= \ln|x+3| - \ln|x+2| + C.
\end{aligned}$$

NOTE: Any expression of the form $\displaystyle\frac{f(x)}{g(x)}$ where

1. $f(x)$ and $g(x)$ are polynomials and $g(x)$ has higher degree than $f(x)$, and
2. $g(x)$ can be factorized as the product of distinct linear factors
$$g(x) = (x-a_1)(x-a_2)\dots(x-a_k)$$

has a *partial fraction expansion* of the form
$$\frac{f(x)}{g(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_k}{x-a_k},$$
where $A_1,\ A_2,\dots,A_k$ are numbers.

Example 2.3.3 *Determine* $\displaystyle\int \frac{x^3+3x+2}{x+1}\,dx$.

In this example the degree of the numerator exceeds the degree of the denominator, so first apply long division to find the quotient and remainder upon dividing $x^3 + 3x + 2$ by $x = 1$.

We find that the quotient is $x^2 - x + 4$ and the remainder is -2. Hence

$$\frac{x^3 + 3x + 2}{x + 1} = x^2 - x + 4 + \frac{-2}{x + 1}.$$

Thus

$$\int \frac{x^3 + 3x + 2}{x + 1}\, dx = \int x^2 - x + 4\, dx - 2\int \frac{1}{x + 1}\, dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x - 2\ln|x + 1| + C.$$

NOTE: In the above example we had $\frac{f(x)}{g(x)}$ with $f(x)$ of greater degree than $g(x)$. In such cases we can always write

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)},$$

where the polynomials $q(x)$ and $r(x)$ are the quotient and remainder respectively on dividing $f(x)$ by $g(x)$, and the degree of $r(x)$ is less than that of $g(x)$.

Example 2.3.4 *Determine* $\int \frac{x + 1}{(2x + 1)^2(x - 2)}\, dx.$

In this case the denominator has a repeated linear factor $2x + 1$. It is necessary to include both $\frac{A}{2x + 1}$ and $\frac{B}{(2x + 1)^2}$ in the partial fraction expansion. We have

$$\frac{x + 1}{(2x + 1)^2(x - 2)} = \frac{A}{2x + 1} + \frac{B}{(2x + 1)^2} + \frac{C}{x - 2}.$$

Then

$$\frac{x + 1}{(2x + 1)^2(x - 2)} = \frac{A(2x + 1)(x - 2) + B(x - 2) + C(2x + 1)^2}{(2x + 1)^2(x - 2)}.$$

This means that the polynomials $x + 1$ and $A(2x + 1)(x - 2) + B(x - 2) + C(2x + 1)^2$ are equal, and therefore have the same value when x is replaced by any real number.

$$\begin{array}{rll} x = 2: & 3 = C(5)^2 & C = \frac{3}{25} \\[2ex] x = -\frac{1}{2}: & \frac{1}{2} = B\left(-\frac{5}{2}\right) & B = -\frac{1}{5} \\[2ex] x = 0: & 1 = A(1)(-2) + B(-2) + C(1)^2 & A = -\frac{6}{25} \end{array}$$

Thus

$$\frac{x + 1}{(2x + 1)^2(x - 2)} = \frac{-6/25}{2x + 1} + \frac{-1/5}{(2x + 1)^2} + \frac{3/25}{x - 2}$$

and

$$\int \frac{x + 1}{(2x + 1)^2(x - 2)}\, dx = -\frac{6}{25}\int \frac{1}{2x + 1}\, dx - \frac{1}{5}\int \frac{1}{(2x + 1)^2}\, dx + \frac{3}{25}\int \frac{1}{x - 2}\, dx.$$

Call the three integrals on the right above $I_1,\ I_2,\ I_3$ respectively.

- I_1 : Let $u = 2x + 1,\ du = 2dx,\ dx = \frac{1}{2}du.$
  $\int \frac{1}{2x+1}\, dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|(+C_1) = \frac{1}{2}\ln|2x + 1|(+C_1).$

- I_2 : Let $u = 2x + 1,\ du = 2dx,\ dx = \frac{1}{2}du$.

  $\int \frac{1}{(2x+1)^2}\,dx = \frac{1}{2}\int u^{-2}\,du = -\frac{1}{2}u^{-1}(+C_2) = -\frac{1}{2(2x+1)}(+C_2)$.

- I_3 : $\int \frac{1}{x-2}\,dx = \ln|x-2|(+C_3)$.

Thus

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{3}{25}\ln|2x+1| + \frac{1}{10(2x+1)} + \frac{3}{25}\ln|x-2| + C.$$

2.4 Integration by parts - reversing the product rule

In this section we discuss the technique of "integration by parts", which is essentially a reversal of the product rule of differentiation.

Example 2.4.1 *Find* $\int x\cos x\,dx$.

There is no obvious substitution that will help here.
How could $x\cos x$ arise as a derivative?
Well, $\cos x$ is the derivative of $\sin x$. So, if you were differentiating $x\sin x$, you would get $x\cos x$ but according to the product rule you would also get another term, namely $\cos x$. Thus

$$\begin{aligned}\frac{d}{dx}(x\sin x) &= x\cos x+\sin x\\ \Longrightarrow \frac{d}{dx}(x\sin x)-\sin x &= x\cos x.\end{aligned}$$

Note that $\sin x=\frac{d}{dx}(-\cos x)$. So

$$\frac{d}{dx}(x\sin x)-\frac{d}{dx}(-\cos x)=x\cos x \Longrightarrow \frac{d}{dx}(x\sin x+\cos x)=x\cos x.$$

CONCLUSION: $\int x\cos x\,dx = x\sin x+\cos x+C$.

What happened in this example was basically that the product rule was reversed. This process can be managed in general as follows. Recall from differential calculus that if u and v are expressions involving x, then

$$(uv)'=u'v+uv'.$$

Suppose we integrate both sides here with respect to x. We obtain

$$\int(uv)'\,dx=\int u'v\,dx+\int uv'\,dx \Longrightarrow uv=\int u'v\,dx+\int uv'\,dx.$$

This can be rearranged to give the *Integration by Parts Formula* :

$$\int uv'\,dx = uv-\int u'v\,dx.$$

Strategy : when trying to integrate a product, assign the name u to one factor and v' to the other. Write down the corresponding u' (the derivative of u) and v (an antiderivative of v').

The integration by parts formula basically allows us to exchange the problem of integrating uv' for the problem of integrating $u'v$ - which might be easier, if we have chosen our u and v' in a sensible way.

Here is the first example again, handled according to this scheme.

Example 2.4.2 *Use the integration by parts technique to determine* $\int x\cos x\,dx$.

SOLUTION: Write

$$\begin{array}{ll} u=x & v'=\cos x\\ u'=1 & v=\sin x\end{array}$$

Then

$$\begin{aligned}\int x\cos x\,dx &= \int uv'\,dx = uv-\int u'v\,dx\\ &= x\sin x-\int 1\sin x\,dx\\ &= x\sin x+\cos x+C.\end{aligned}$$

NOTE: We could alternatively have written $u = \cos x$ and $v' = x$. This would be less successful because we would then have $u' = -\sin x$ and $v = \frac{x^2}{2}$, which looks worse than v'. The integration by parts formula would have allowed us to replace

$$\int x \cos x \, dx \quad \text{with} \quad \int \frac{x^2}{2} \sin x \, dx,$$

which is not an improvement.

So it matters which component is called u and which is called v'.

Example 2.4.3 *To determine* $\int \ln x \, dx$.

SOLUTION: Let $u = \ln x$, $v' = 1$. Then $u' = \frac{1}{x}$, $v = x$.

$$\begin{aligned} \int \ln x \, dx &= \int uv' \, dx = uv - \int u'v \, dx \\ &= x \ln x - \int \frac{1}{x} x \, dx \\ &= x \ln x - x + C. \end{aligned}$$

NOTE: Example 2.4.3 shows that sometimes problems which are not obvious candidates for integration by parts can be attacked using this technique.

Sometimes two applications of the integration by parts formula are needed, as in the following example.

Example 2.4.4 *To evaluate* $\int x^2 e^x \, dx$.

SOLUTION: Let $u = x^2$, $v' = e^x$. Then $u' = 2x$, $v = e^x$.

$$\begin{aligned} \int x^2 e^x \, dx &= \int uv' \, dx = uv - \int u'v \, dx \\ &= x^2 e^x - \int 2x e^x \, dx \\ &= x^2 e^x - 2 \int x e^x \, dx. \end{aligned}$$

Let $I = \int x e^x \, dx$.

To evaluate I apply the integration by parts formula a second time.

$$\begin{array}{ll} u = x & v' = e^x \\ u' = 1 & v = e^x. \end{array}$$

Then $I = \int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C$. Finally

$$\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C.$$

The next example shows another mechanism by which a second application of the integration by parts formula can succeed where the first is not enough.

Example 2.4.5 *Determine* $\int e^x \cos x \, dx$.

Let

$$\begin{array}{ll} u = e^x & v' = \cos x \\ u' = e^x & v = \sin x. \end{array}$$

Then

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

For $\int e^x \sin x \, dx$: Let

$$\begin{array}{ll} u = e^x & v' = \sin x \\ u' = e^x & v = -\cos x. \end{array}$$

Then

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx,$$

and

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx \right)$$

$$\Longrightarrow 2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C$$

$$\Longrightarrow \int e^x \cos x \, dx = \frac{1}{2}(e^x \sin x + e^x \cos x) + C$$

Finally, an example of a definite integral evaluated using the integration by parts technique.

Example 2.4.6 *Evaluate* $\int_0^1 (x+3)e^{2x} \, dx$.

SOLUTION: Write

$$\begin{array}{ll} u = x + 3 & v' = e^{2x} \\ u' = 1 & v = \frac{1}{2}e^{2x} \end{array}$$

Then

$$\begin{aligned}
\int_0^1 (x+3)e^{2x} \, dx &= \int uv' \, dx = (uv)|_0^1 - \int_0^1 u'v \, dx \\
&= \frac{x+3}{2} e^{2x} \Big|_0^1 - \frac{1}{2} \int_0^1 e^{2x} \, dx \\
&= \frac{x+3}{2} e^{2x} \Big|_0^1 - \frac{1}{2} \times \frac{1}{2} e^{2x} \Big|_0^1 \\
&= \frac{4}{2} e^2 - \frac{3}{2} e^0 - \frac{1}{4} e^2 + \frac{1}{4} e^0 \\
&= \frac{7}{4} e^2 - \frac{5}{4}.
\end{aligned}$$

Chapter 3

Areas and Volumes

3.1 Areas of Regions of the Plane

Recall that

$$\int_a^b f(x)\,dx$$

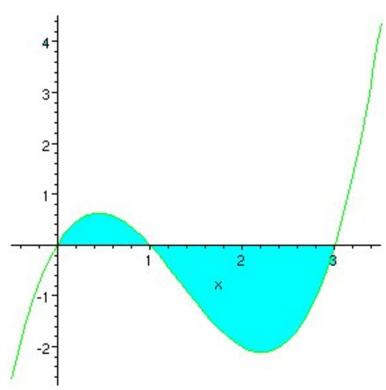
is the area enclosed between the graph $y = f(x)$ and the x-axis between $x = a$ and $x = b$, where area below the x-axis is counted negatively. This fact can be used to calculate (genuine) areas of regions in the plane.

Example 3.1.1 *Find the total area of the region between the x-axis and the graph $y = x^3 - 4x^2 + 3x$, $0 \leqslant x \leqslant 3$.*

SOLUTION: This may not necessarily be just $\int_0^2 x^3 - 4x^2 + 3x\,dx$, since the graph may lie below the x-axis on part or all of the interval $[0, 3]$.
First find the zeroes of $x^3 - 4x^2 + 3x$ (i.e. the points where the graph $y = x^3 - 4x^2 + 3x$ crosses the x-axis):

$$x^3 - 4x^2 + 3x = x(x^2 - 4x + 3) = x(x-1)(x-3) : \text{ zeroes at } x = 0, 1, 3.$$

Now sketch the graph of $f(x)$ in the relevant region.



We require the total area enclosed between the graph and the x-axis. This is given by $A_1 + A_2$, where

$$\begin{aligned} A_1 &= \int_0^1 x^3 - 4x^2 + 3x\,dx, \\ &= \frac{x^4}{4} - 4\frac{x^3}{3} + 3\frac{x^2}{2}\Big|_0^1 \\ &= \frac{1}{4} - \frac{4}{3} + \frac{3}{2} \\ &= \frac{5}{12}. \end{aligned}$$

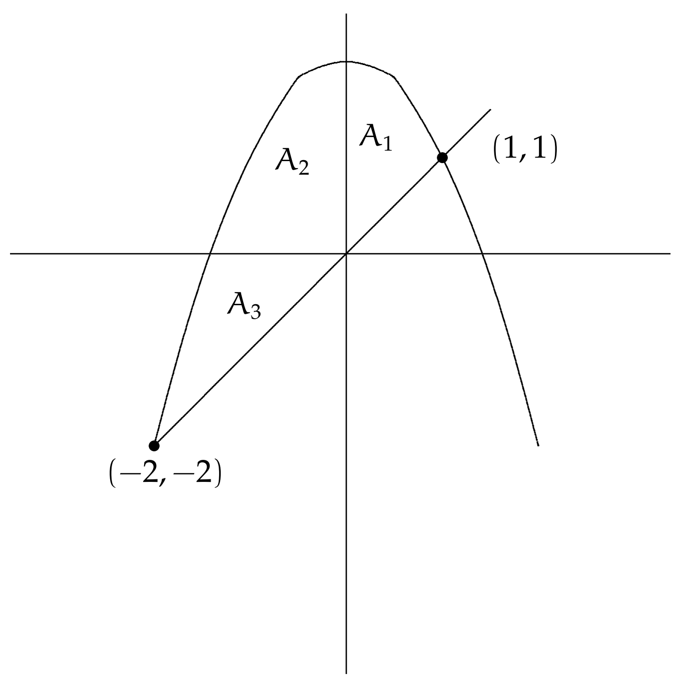
and

$$\begin{aligned} A_2 &= -\int_1^3 x^3 - 4x^2 + 3x\,dx \\ &= \frac{x^4}{4} - 4\frac{x^3}{3} + 3\frac{x^2}{2}\;|_1^3 \\ &= -\left(\left(\frac{81}{4} - 36 + \frac{27}{2}\right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right)\right) \\ &= \frac{8}{3}. \end{aligned}$$

So the total area is $\frac{5}{12} + \frac{8}{3} = \frac{37}{12}$.

Example 3.1.2 *Find the area enclosed by the curves $y = 2 - x^2$ and $y = x$.*

SOLUTION: Draw a rough sketch to get a picture of the region involved.



We need the points of intersection of the two curves. They intersect when

$$2 - x^2 = x \Longrightarrow x^2 + x - 2 = 0 \Longrightarrow (x+2)(x-1) = 0.$$

The points of intersection are $(-2,-2)$ and $(1,1)$.
The area required is $A_1 + A_2 + A_3$, where

$$\begin{aligned} A_1 &= \int_0^1 2 - x^2\, dx - \int_0^1 x\, dx \\ &= \int_0^1 2 - x^2 - x\, dx. \end{aligned}$$

NOTE: The integral $\int_0^1 2 - x^2\, dx$ would give us *all* the area under the graph $y = 2 - x^2$, between $x = 0$ and $x = 1$. To get A_1 we need to subtract the part of this area that is below the line $y = -x$; this is $\int_0^1 x\, dx$.

$$A_2 = \int_{-\sqrt{2}}^0 2 - x^2\, dx$$

NOTE: The integral $\displaystyle\int_{-\sqrt{2}}^0 2 - x^2\, dx$ is exactly the area labelled A_2 in the diagram.

$$\begin{aligned} A_3 &= -\int_{-2}^0 x\, dx - (-\int_{-2}^{-\sqrt{2}} 2 - x^2\, dx \\ &= \int_{-2}^{-\sqrt{2}} 2 - x^2\, dx - \int_{-2}^0 x\, dx. \end{aligned}$$

NOTE: The expression $-\int_{-2}^0 x\, dx$ would give us *all* the area enclosed between the line $y = -x$ and the x-axis, between $x = -2$ and $x = 0$. From the diagram we can see that in order to calculate the area A_3 we need to subtract $\left(-\displaystyle\int_{-2}^{-\sqrt{2}} 2 - x^2\, dx\right)$ from this quantity.

We can observe that

$$A_2 + A_3 = \int_{-\sqrt{2}}^0 2 - x^2\, dx + \int_{-2}^{-\sqrt{2}} 2 - x^2\, dx - \int_{-2}^0 x\, dx = \int_{-2}^0 2 - x^2 - x\, dx.$$

Hence the total area is

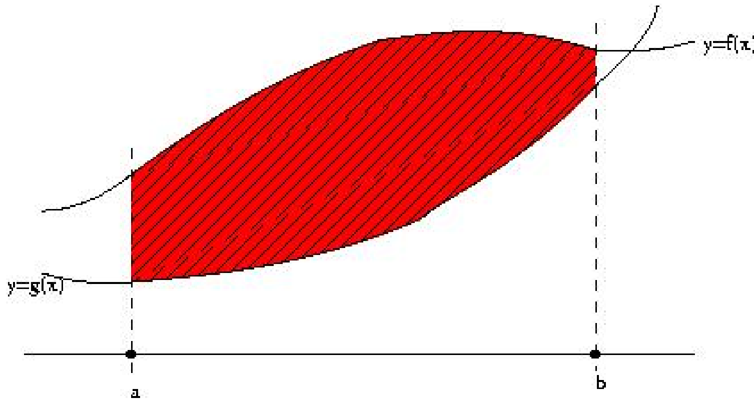
$$A_1 + A_2 + A_3 = \int_0^1 2 - x^2 - x\, dx + \int_{-2}^0 2 - x^2 - x\, dx = \int_{-2}^1 2 - x^2 - x\, dx.$$

Evaluating this definite integral results in the answer $\dfrac{59}{6}$.

Example 3.1.2 above relates to the following fact : Suppose that f and g are functions of x for which $f(x) \geqslant g(x)$ for every x in the interval $[a, b]$ (so the graph $y = f(x)$ sits above $y = g(x)$ on the interval $[a, b]$). Then the area enclosed between the graphs $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ is

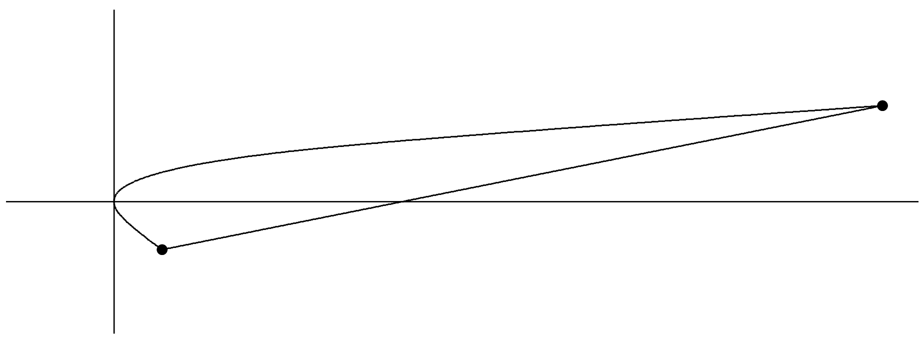
$$\int_a^b f(x) - g(x)\, dx.$$

The area shaded in the diagram below is $\displaystyle\int_a^b f(x) - g(x)\, dx$.



Example 3.1.3 *Find the area enclosed by the graph* $x = y^4$ *and the line through the points* $(1, -1)$ *and* $(16, 2)$.

SOLUTION: The following diagram shows the graph $x = y^4$ and the line joining the points $(1, -1)$ and $(16, 2)$.



COMMENT: We *could* tackle this problem in the same way as Example 3.1.2, by dividing the area into regions and interpreting the regions separately as definite integrals. This would be fine; but there is a slight difficulty caused by the problem of expressing the equation $x = y^4$ in the form $y = f(x)$. The top half of this curve has equation $y = \sqrt[4]{x}$, and the bottom half of it has equation $y = -\sqrt[4]{x}$. Moreover, integrating $\sqrt[4]{x}$ is not that easy; either is dividing the region into sub-regions.

Another way to tackle this problem is to interpret the area as an integral over y rather than x. First figure out the equation of the line through $(1, -1)$ and $(16, 2)$:
its slope is $\dfrac{2-(-1)}{16-1} = \dfrac{3}{15} = \dfrac{1}{5}$.
So its equation has the form $y = \frac{1}{5}x + C$; to get a value for C substitute $x = 1,\ y = -1$. Then

$$-1 = \frac{1}{5}(1) + C \Longrightarrow C = -\frac{6}{5}.$$

The equation of the line is

$$y = \frac{1}{5}x - \frac{6}{5} \text{ or } x = 5y + 6.$$

The area enclosed between the y-axis and the line $x = 5y + 6$, between $y = -1$ and $y =$ is given by

$$\int_{y=-1}^{y=2} 5y + 6\, dy.$$

To get the area A we need to subtract from this the area between the graph $x = y^4$ and the y-axis, which is given by

$$\int_{y=-1}^{y=2} y^4 \, dy.$$

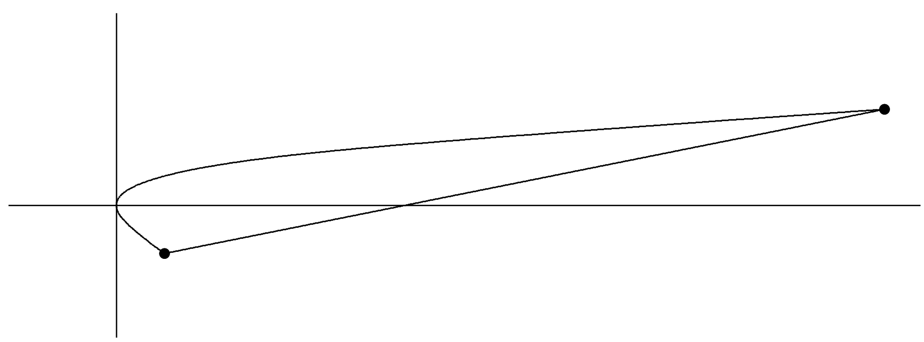
Hence the area A is given by

$$A = \int_{y=-1}^{y=2} 5y + 6 - y^4 \, dy = \left(5\frac{y^2}{2} + 6y - \frac{y^5}{5}\right)\Bigg|_{-1}^{2}.$$

Thus $A = \dfrac{189}{10}$. □

NOTE ON EXAMS/ASSESSMENTS: The commentary above is intended as an explanation of how we might approach a problem like this, which I hope that some students might find helpful in thinking about how to tackle other problems of a similar nature. I would not expect this much commentary in a response to an exam question (although I wouldn't object!). Some students indicated in response to the recent survey on this course that they would appreciate more "sample exam answers". From now on I will try to include examples of what I would consider to be appropriate for an exam answer, alongside the more detailed explanations that are typical of these lecture notes.
So, if Example 3.1.3 appeared on an exam, the following would be a sufficient response and would get full marks.
By the way: I *do* expect answers to exam questions to be written in sentences and to show a clear line of reasoning!

ANSWER: The following diagram shows the region :



The equation of the line through the points $(1, -1)$ and $(16, 2)$ is $x = 5y + 6$.
The required area is given by

$$\int_{y=-1}^{y=2} 5y + 6 - y^4 \, dy = \left(5\frac{y^2}{2} + 6y - \frac{y^5}{5}\right)\Bigg|_{-1}^{2} = \frac{189}{10}.$$

3.2 Volumes of Revolution

In this section we consider the problem of calculating the volume of the solid object obtained by rotating a two-dimensional region around an axis. Examples of shapes that can be obtained in this manner include all three dimensional objects with rotational symmetry of some sort, such as cylinders, cones, spheres, ellipsoids (squashed spheres), and "cylinder-like" shapes with curved surfaces.

Example 3.2.1 *Find the volume of the solid obtained by rotating the region bounded by the curves* $y = x^2, y = 0, x = 0$ *and* $x = 1$ *about the* x*-axis.*

SOLUTION: The following pictures show the region in question and the volume of revolution.

The basic strategy is to take vertical cross-sections of the solid object, perpendicular to the x-axis. These are circular discs - so we know how to compute their area. *The volume of the solid of revolution is the integral of the cross-sectional area, as* x *goes from* 0 *to* 1.

Choose a point x in the interval 0 to 1. The radius of the circular cross-section at x is $r = x^2$. *This is because the upper boundary of the region is the curve* $y = x^2$.

So : the area of the cross-section at x is given by

$$\pi r^2 = \pi(x^2)^2 = \pi x^4.$$

To calculate the volume V of the solid of revolution, we need to integrate this cross-sectional area from $x = 0$ to $x = 1$. Then

$$V = \int_0^1 \pi x^4 \, dx = \pi \frac{x^5}{5}\bigg|_0^1 = \frac{\pi}{5}.$$

NOTE : If this was an exam question, the following would be sufficient as an answer and would get full marks.

- *A sketch showing the region* (this is not strictly necessary but most people find it much easier to tackle problems of this sort if they have a diagram available. An answer that doesn't include a diagram but is otherwise fully correct would get full marks).
- The radius of the cross section at x is x^2. The area of this cross section is πx^4.
  The volume is given by
  $$V = \int_0^1 \pi x^4 \, dx = \pi \frac{x^5}{5}\bigg|_0^1 = \frac{\pi}{5}.$$

GENERAL FORMULA: Suppose the region bounded by the graph $y = f(x)$ and the lines $x = a$ and $x = b$ is rotated about the x-axis. The volume generated is

$$\int_a^b \pi(f(x))^2 \, dx.$$

Note that $\pi(f(x))^2$ is the area of the cross section at x perpendicular to the x-axis.
Alternative Version: If the region bounded by the graph $x = g(y)$ and the lines $y = a$ and $y = b$ is rotated about the y-axis, the volume generated is

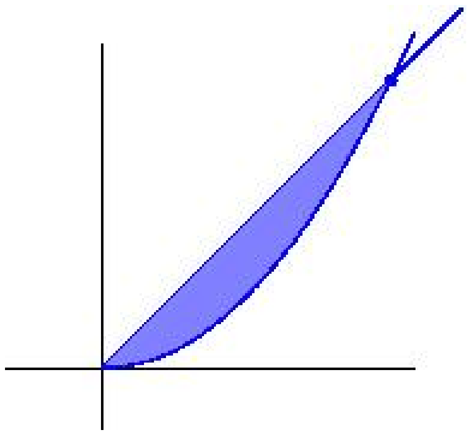
$$\int_a^b \pi(g(y))^2 \, dy.$$

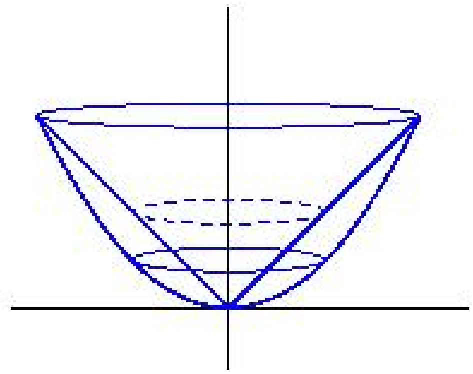
Here $\pi(g(y))^2$ is the area of the cross-section at y, perpendicular to the y-axis.

Example 3.2.2 *Find the volume* V *obtained by rotating the region bounded by* $y = x^2$ *and* $y = x$ *about the* y*-axis.*

SOLUTION:

The diagram below shows the region under consideration and the three-dimensional object obtained by rotating it about the y-axis. This object is a bowl with a parabaloid as the exterior surface and a cone as the interior surface.





To calculate the volume V of the solid of revolution, we integrate the horizontal cross-sectional area from $y = 0$ to $y = 1$. This means that we need to describe the horizontal cross-sectional area at height y, in terms of y.
At height y, the horizontal cross-section is a "washer" whose outer radius is the corresponding value of x on the curve $y = x^2$, and whose inner radius is the corresponding value of x on th curve $y = x$.
Thus the horizontal cross-section at height y is a washer whose outer radius is $\sqrt{y}$ and whose inner radius is y. The area of this washer is

$$\pi(\sqrt{y})^2 - \pi y^2 = \pi(y - y^2).$$

The volume V is the integral of this expression from $y = 0$ to $y = 1$, i.e.

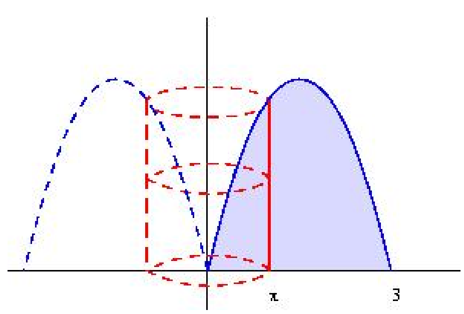
$$V = \int_0^1 \pi(y - y^2)\,dy = \pi\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}.$$

Example 3.2.3 *Find the volume V generated when the region enclosed by the x-axis and the graph $y = 3x - x^2$ is rotated about the y-axis.*

SOLUTION: For this example the horizontal cross sections are unwieldy to manage. When the equation of the parabola is rewritten to express x in terms of y, the resulting description is not very easy to handle.

Fortunately we have another approach, known as the "method of cylindrical shells" which is more convenient for this example.

The picture below shows the region in question, shaded in blue on the right hand side of the diagram.



When this region is rotated about the y-axis, think of the shape traced out by the red vertical line segment. It traces out a "cylindrical shell" - a hollow cylinder of radius x and height $3x - x^2$. The surface area of this cylindrical shell is given by

$$2\pi rh = 2\pi x(3x - x^2) = 2\pi(3x^2 - x^3).$$

The volume V of the solid of revolution can be calculated by integrating the surface area of this cylindrical shell, from $x = 0$ to $x = 3$. Thus

$$V = \int_0^3 2\pi(3x^2 - x^3)\,dx = 2\pi\left(x^3 - \frac{x^4}{4}\right)\Bigg|_0^3 = 2\pi\left(27 - \frac{81}{4}\right) = \frac{27}{2}\pi.$$

Chapter 4

Differential Equations in Science

4.1 Differential Equations

A *differential equation* is an equation that involves some quantity and its derivative, or possibly its higher-order derivatives. Many physical processes are described by differential equations. For example if an object is falling freely under gravity it is accelerating at $9.8\text{m}/\text{s}^2$, and so its speed v satisfies the differential equation

$$\frac{dv}{dt} = 9.8.$$

Other physical phenomena that are described by differential equations include the temperature of a warm object that is placed in a cool environment. *Newton's Law of Cooling* states that the rate at which the temperature is decreasing at any instant is proportional to the difference between the temperature of the object and its surroundings, *at that instant*.

Another example is offered by *exponential growth* patterns. Suppose that a colony of bacteria has unlimited resources in which to grow. Suppose that the number of organisms in the colony doubles after five days. This might mean, for example, that every organism splits in two once every five days. We can expect the population to double again in another five days, and we can expect this pattern to continue until resources become scarce. The number of new bacteria produced in a given five-day period is basically the number that were present at the start of this five-day period. The *rate* at which the population is growing is the number of new organisms that are being produced per unit of time. According to this model, this rate is not constant but at any moment it is proportional to the population size. In terms of calculus this is expressed as follows. The population P is a function of time (t). At any given moment, the rate at which the population is increasing is directly proportional to the number of organisms present at that moment. Thus

$$\frac{dP}{dt} = \alpha P(t),$$

for a (positive) constant α. This is a differential equation satisfied by the growth of the population of bacteria.

NOTE: This growth pattern is called *exponential growth*, for a reason that we will see shortly. Another example of something that exhibits exponential growth is a sum of money on deposit, accumulating interest continuously. If the constant α in the equation above is negative, then P is *decreasing* according to an *exponential decay* pattern. Exponential decay occurs for example in radioactive substances, where the number of radioactive nuclei decays at a rate that is proportional to the present quantity.

Other examples of differential equations which describe physical phenomena include the *heat equation*, which describes heat flow in space and time, the *wave equation* which describes the propagation of waves, and the *logistic equation* which describes the growth of populations constrained by resource limitations. We will look at some of these in this section. The exam questions on this topic will be closely related to the examples in this section.

Example 4.1.1 *The half-life of the carbon–14 isotope is 5700 years. A fossilized fragment of a tree trunk found in 2008 contained 17% of the carbon–14 content found in living tissue. Estimate when the tree died.*

NOTE: The *half-life* of carbon–14 means the time taken for half of the radioactive nuclei in a sample to decay (to nitrogen–14). This time does *not* depend on the size of the original sample, because of the pattern of exponential decay that prevails. This is similar to the "bacterial population growth" example above.
SOLUTION: Let $x(t)$ be the quantity of carbon–14 present at time t, and suppose that the tree died at time $t = 0$. We know that $x(t)$ satisfies the differential equation

$$\frac{dx}{dt} = ax,$$

where a is a (negative) constant.

The first step is to re-organize this equation to obtain a direct description of how x depends on t. We can rewrite the above equation and integrate both sides as follows :

$$\frac{dx}{dt} = ax \Longrightarrow \frac{dx}{x} = a\,dt \Longrightarrow \ln x = at + C.$$

Now take exponentials on both sides to obtain

$$e^{\ln x} = e^{at+C} \Longrightarrow x(t) = e^{at}e^{C} \Longrightarrow x(t) = Ae^{at},$$

where we give the name A to the positive constant e^C.
This formula involves two unknown constants, A and a. Note that

$$x(0) = Ae^0 = A,$$

so A is the quantity of carbon-14 present at time $t = 0$. In order to find the value of a, we use our knowledge that the half-life of carbon–14 is 5700 years. This means that after 5700 years, the amount of carbon 14 present is half of the original A. Now

$$Ae^{5700a} = 0.5A \Longrightarrow e^{5700a} = 0.5 \Longrightarrow 5700a = \ln 0.5 \Longrightarrow a = \frac{\ln 0.5}{5700}$$

We need to know the time T for which $x(T) = 0.17A$. This means

$$\begin{aligned} Ae^{\frac{\ln 0.5}{5700}T} &= 0.17A \\ \Longrightarrow e^{\frac{\ln 0.5}{5700}T} &= 0.17 \\ \Longrightarrow \frac{\ln 0.5}{5700}T &= \ln 0.17 \\ \Longrightarrow T &= \frac{\ln 0.17 \times 5700}{\ln 0.5} \\ &\approx 14571 \end{aligned}$$

So the tree had been dead for 14571 years - it died around 12500 BC.

Example 4.1.2 *The half-life of the isotope neptunium–239 is 2.36 days. An experiment initially involves a 12g sample of neptunium–239. The sample will no longer be useful for the experiment after 70% of the radioactive nuclei present have decayed. For how many days can the experiment proceed?*

NOTE: The solution below has the style and level of detail that would be expected in an "ideal" exam answer to this question.
SOLUTION: Let $x(t)$ be the mass in grammes of neptunium present at time t (measured in days), and let $t = 0$ at the start of the experiment. We need to find the time T when $x(T) = 0.3 \times 12 = 3.6$. We know that $x(0) = 12$ and as in Example 4.1.1 we know that

$$x(t) = Ae^{at}$$

where A and a are constants.
Now $x(0) = 12 = Ae^0$, so $A = 12$ and $x(t) = 12e^{at}$.
To find a, use the fact that $x(2.36) = 6$. Thus

$$x(2.36) = 12e^{2.36a} = 6 \Longrightarrow e^{2.36a} = 0.5 \Longrightarrow 2.36a = \ln 0.5 \Longrightarrow a = \frac{\ln 0.5}{2.36}.$$

Thus $x(t) = 12e^{\frac{\ln 0.5}{2.36}t}$. We need the time T for which $x(T) = 3.6$. This means

$$12e^{\frac{\ln 0.5}{2.36}T} = 3.6 \Longrightarrow e^{\frac{\ln 0.5}{2.36}T} = \frac{3.6}{12} = 0.3 \Longrightarrow \frac{\ln 0.5}{2.36}T = \ln 0.3.$$

Thus

$$T = \frac{2.36 \times \ln 0.3}{\ln 0.5} \approx 4.1.$$

The experiment can continue for just over four days.

Population Growth - the Logistic Differential Equation

In the earlier discussion about growth of a culture of bacteria, we assumed that the population of bacteria grows at a rate proportional to its size and hence has an exponential growth pattern. This may be reasonable as long as there is no scarcity of resources, but it cannot remain realistic indefinitely. A more realistic mathematical model of population growth is provided by the *logistic differential equation*, which works as follows.

Let $y(t)$ be the population (of humans, birds, fish, trees, whatever) at time t. Assume that the system can sustain a maximum population of L. Thus at time t the room for population growth is given by $L - y(t)$. We assume that the rate of population growth is proportion to the product of the present size of the population (which basically measures the opportunity for reproduction) and the room for growth. So it is proposed that y satisfies a differential equation of the form

$$\frac{dy}{dt} = ay(L - y),$$

for a constant a. This equation is called the *logistic* differential equation.
NOTE: The validity of this model and the appropriate values of L and a would be determined by experimental data and knowledge of the ecological situation. The presence of the term $L - y$ means that the population growth will slow down when $L - y$ gets small (i.e. when the actual population y nears the maximum sustainable population L).

Example 4.1.3 *The population y of starfish (in hundreds of starfish) in a coastal bay is modelled by the logistic equation*

$$\frac{dy}{dt} = 0.0003y(2000 - y),$$

where time t is measured in years. The present starfish population is believed to be 80000 starfish. Use the logistic equation to predict the starfish population in two years.

SOLUTION: We need to *solve* the logistic equation to find a formula that expresses y in terms of t. Suppose that $t = 0$ now. So $y(0) = 800$. This is the *initial condition*. We want to determine $y(2)$. The first step is to *separate the variables* in the differential equation : bring all appearances of y to the left and all appearances of t to the right. Then

$$\frac{dy}{dt} = 0.0003y(2000 - y) \Longrightarrow \frac{dy}{y(2000 - y)} = 0.0003\,dt \Longrightarrow \int \frac{dy}{y(2000 - y)} = \int 0.0003\,dt.$$

For the integral on the left, use a partial fraction expansion (see Section 2.3). Write

$$\frac{1}{y(2000 - y)} = \frac{A}{y} + \frac{B}{2000 - y} \Longrightarrow 1 = A(2000 - y) + By.$$

Setting $y = 2000$ gives $B = \frac{1}{2000}$.
Setting $y = 0$ gives $A = \frac{1}{2000}$.
Thus

$$\frac{1}{y(2000-y)} = \frac{1}{2000y} + \frac{1}{2000(2000-y)} = \frac{1}{2000}\left(\frac{1}{y} + \frac{1}{2000-y}\right).$$

Now

$$\int \frac{dy}{y(2000-y)} = \frac{1}{2000}\left(\int \frac{dy}{y} + \int \frac{dy}{2000-y}\right) = \frac{1}{2000}(\ln y - \ln(2000-y)) + \text{constant}.$$

Note that $\ln y - \ln(2000-y) = \ln\left(\frac{y}{2000-y}\right)$. Finally

$$\begin{aligned}
\int \frac{dy}{y(2000-y)} &= \int 0.0003\, dt \\
\Longrightarrow \frac{1}{2000}\ln\left(\frac{y}{2000-y}\right) &= 0.0003t + C \\
\ln\left(\frac{y}{2000-y}\right) &= 0.6t + 2000C \\
\Longrightarrow \frac{y}{2000-y} &= e^{0.6t}e^{2000C} = Ae^{0.6t}
\end{aligned}$$

In the last line, the name A was given to the constant e^{2000C}.

Now a formula to describe how y depends on t can be derived as follows:

$$y = 2000Ae^{0.6t} - yAe^{0.6t} \Longrightarrow y = \frac{2000Ae^{0.6t}}{1 + Ae^{0.6t}}.$$

We know that $y(0) = 800$; setting $t = 0$ above gives

$$800 = \frac{2000A}{1+A} \Longrightarrow 1200A = 800,\ A = \frac{2}{3}.$$

Then

$$y(t) = y = \frac{(4000/3)e^{0.6t}}{1 + (2/3)e^{0.6t}}.$$

Finally set $t = 2$ to answer the question :

$$y(2) = \frac{(4000/3)e^{1.2}}{1 + (2/3)e^{1.2}} \approx 1377.$$

So the expected starfish population in two years is 137700.