# Special spaces of matrices

IMS Meeting 2013
NUI Maynooth

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August 27, 2013

# Basic questions

- What is the maximum possible dimension of a linear (or affine) space of (any, or symmetric, or skew-symmetric ...) matrices in $M_n(\mathbb{F})$ in which
  - all (non-zero) elements have the same rank, or
  - the ranks of (non-zero) elements all lie between specified bounds.
- How do examples achieving these bounds arise?

# Example - spaces of nonsingular matrices

Suppose that $A$ and $B$ are invertible matrices in $\mathrm{GL}(n, \mathbb{C})$, and that $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} \det(\lambda A + B) &= \det(A)\det(\lambda I_n + A^{-1}B) \\ \implies \det(\lambda A + B) = 0 &\iff \det(\lambda I_n + A^{-1}B) = 0. \end{aligned}$$

Since $\det(\lambda I_n + A^{-1}B)$ is a polynomial of degree $n$ in $\lambda$, it has a root in $\mathbb{C}$.

### Theorem

- *The maximum possible dimension of a space of nonsingular matrices in $M_n(\mathbb{C})$ is 1.*
- *If $n$ is odd, the maximum possible dimension of a space of nonsingular matrices in $M_n(\mathbb{R})$ is 1.*

# Smaller fields

Suppose $\mathbb{F}$ is a field that admits a field extension $\mathbb{K}$ of degree $n$, so that

$$\dim_{\mathbb{F}}(\mathbb{K}) = n.$$

For each $\alpha \in \mathbb{K}$, define $f_\alpha : \mathbb{K} \to \mathbb{K}$ by

$$f_\alpha(x) = \alpha x.$$

Then $f_\alpha$ is an invertible $\mathbb{F}$-linear transformation of $\mathbb{K}$.
Let $M_\alpha$ be the matrix of this transformation with respect to some specified $\mathbb{F}$-basis of $\mathbb{K}$. Then

$$\alpha \to M_\alpha$$

is an $\mathbb{F}$-linear isomorphism of fields and

$$\{M_\alpha : \alpha \in \mathbb{K}\}$$

is a space of non-singular matrices of dimension $n$ in $M_n(\mathbb{F})$.

# On the other hand ...

If $\{A_1, \ldots, A_k\}$ are linearly independent elements of a space of nonsingular matrices in $M_n(\mathbb{F})$, then the first rows of these matrices must be linearly independent over $\mathbb{F}$. Thus the dimension of a space of nonsingular matrices in $M_n(\mathbb{F})$ cannot exceed $n$.

If $X \subset M_n(\mathbb{F})$ is a space of invertible matrices of dimension $n$, then there is a isomorphism $\phi : \mathbb{F}^n \to X$ of $\mathbb{F}$-vector spaces. Defining a multiplication $\cdot$ on $\mathbb{F}^n$ by $u \cdot v = \phi(u)v$ gives $\mathbb{F}^n$ the structure of a presemifield over $\mathbb{F}$. Hence

**Theorem**

*There exists an n-dimensional subspace of invertible matrices in $M_n(\mathbb{F})$ if and only if there exists a semifield of dimension n over $\mathbb{F}$.*

A semifield satisfies all the axioms of a field except possibly commutativity and associativity of multiplication.

# Nonsingular spaces over $\mathbb{R}$ - the Radon-Hurwitz numbers

All semifields over $\mathbb{R}$ have dimension 1 (real field), 2 (complex field), 4 (quaternion division algebra) or 8 (octonion semifield). For a natural number $n$, define $\rho(n)$ as follows.

- $2^u$ is the highest power of 2 that divides $n$.
- $a$ and $b$ are respectively the quotient and remainder on dividing $u$ by 4.
- $\rho(n) = 8a + 2^b$.

The numbers $\rho(n)$ are the Radon-Hurwitz numbers.

| u | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho(n)$ | 1 | 2 | 4 | 8 | 9 | 10 | 12 | 16 | 17 | 18 | 20 | 24 |

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**Theorem (Adams,Radon,Hurwitz,...)**

*The maximum dimension of a nonsingular subspace of $M_n(\mathbb{R})$ is $\rho(n)$.*

# How to produce a 9-d nonsingular space in $M_{16}(\mathbb{R})$

Let $W$ be a 8-dimensional nonsingular subspace of $M_8(\mathbb{R})$ (constructed from the octonion semifield). For each $A \in W$ and $\lambda \in \mathbb{R}$, define a linear transformation $\tau_{A,\lambda} : \mathbb{R}^8 \oplus \mathbb{R}^8 \to \mathbb{R}^8 \oplus \mathbb{R}^8$ by

$$\tau_{A,\lambda}(x, y) = (Ay + \lambda x, A^T x - \lambda y).$$

Then $\tau_{A,\lambda}$ is an invertible linear transformation of $\mathbb{R}^{16}$, for suppose for some $(x, y) \in \mathbb{R}^8 \oplus \mathbb{R}^8$ that $Ay + \lambda x = A^T x - \lambda y = 0$. Then

$$\begin{aligned} x^T Ay &= -\lambda x^T x \\ x^T Ay &= \lambda y^T y \\ \Longrightarrow \lambda(x^T x + y^T y) = 0 &\Longrightarrow \lambda = 0. \end{aligned}$$

Then $A$ is singular and so $A = 0$. Hence $\{\tau_{A,\lambda} : \; A \in W, \; \lambda \in \mathbb{R}\}$ is (isomorphic to) a 9-dimensional nonsingular subspace of $M_{16}(\mathbb{R})$.

# Vector fields on spheres (Adams, 1962)

$S^{n-1} = \{x \in \mathbb{R}^n : ||x|| = 1\}$: the $(n-1)$-sphere.

A (continuous) vector field on $S^{n-1}$ is a (continuous) mapping $\phi : S^{n-1} \to \mathbb{R}^n$ with the property that $v \cdot \phi(v) = 0$ for all $v \in S^{n-1}$.

Vector fields $\phi_1, \phi_2, \ldots, \phi_k$ on $S^{n-1}$ are called linearly independent if $\{\phi_1(v), \phi_2(v), \ldots, \phi_k(v)\}$ is a linearly independent subset of $\mathbb{R}^n$ for every $v \in S^{n-1}$.

Question What is the maximum number of linearly independent continuous vector fields on $S^{n-1}$?

**Theorem (Adams, 1962)**

*The answer is* $\rho(n) - 1$.

# The connection with nonsingular spaces

Suppose that $\{A_1, \ldots, A_{\rho(n)}\}$ is a basis for a nonsingular subspace $X$ of $M_n(\mathbb{R})$. Let $X'$ denote the subspace spanned by $A_2, \ldots, A_{\rho(n)}$, so $\dim X' = \rho(n) - 1$.

- If $A \in X'$ and $A \neq 0$, note that $A_1^{-1}A$ has no real eigenvalue.
- For $i = 2, \ldots, \rho(n)$, write $B_i = A_1^{-1}A_i$. Define vector fields $\phi_2, \ldots, \phi_{\rho(n)}$ on $S^{n-1}$ by

$$\phi_i(v) = \text{proj}_{v^\perp} B_i(v).$$

- Then these $\phi_i$ are linearly independent vector fields on $S^{n-1}$. Suppose for some $v \in S^{n-1}$ and $c_i \in \mathbb{R}$ that

$$c_2\phi_2(v) + \cdots + c_{\rho(n)}\phi_{\rho(n)}(v) = 0.$$

Then $v$ is an eigenvector of $A_1^{-1}A$, where $A = (c_2A_2 + \cdots + c_{\rho(n)}A_{\rho(n)}) \in X'$, hence $A = 0$ and each $c_i = 0$.

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# The case of affine spaces

**Theorem (Meshulam 1989; Quinlan 2011; McTigue & Quinlan 2011; de Seguins Pazzis 2012; ...)**

*For any field $\mathbb{F}$, the maximum possible dimension of an affine subspace of $M_n(\mathbb{F})$ in which every element is nonsingular is $\frac{n(n-1)}{2}$.*

## Examples

1. $I_n + SUT_n(\mathbb{F})$, the set of upper triangular matrices having 1 in all diagonal positions.
2. If $\mathbb{F}$ is a formally real field (e.g. $\mathbb{R}$), $I_n + A_n(\mathbb{F})$, where $A_n(\mathbb{F}) = \{B \in M_n(\mathbb{F}) : B^T = -B\}$ is the space of skew-symmetric matrices.

# Some related theorems

Definiton For a linear subspace $X$ of $M_{m\times n}(\mathbb{F})$, define $X^\perp$ by

$$X^\perp = \{B \in M_{n\times m}(\mathbb{F}) : \text{trace}(AB) = 0 \ \forall\ A \in X\}.$$

Then $X^\perp$ is a linear space and $\dim(X) + \dim(X^\perp) = mn$.

Note For a linear subspace $X$ of $M_n(\mathbb{F})$, the affine subspace $I_n + X$ consists of nonsingular matrices if and only if no element of $X$ possesses a non-zero eigenvalue in $\mathbb{F}$.

# Some related theorems

Definiton For a linear subspace $X$ of $M_{m\times n}(\mathbb{F})$, define $X^{\perp}$ by

$$X^{\perp} = \{B \in M_{n\times m}(\mathbb{F}) : \mathrm{trace}(AB) = 0 \ \forall \ A \in X\}.$$

Then $X^{\perp}$ is a linear space and $\dim(X) + \dim(X^{\perp}) = mn$.

**Theorem (Duality Theorem, Version 1)**

*Every element of the affine space $I_n + X$ is non-singular if and only if no element of $X$ has a non-zero eigenvalue in $X$, if and only if every non-zero vector in $\mathbb{F}^n$ occurs as the rowspace of some element of non-zero trace in $X^{\perp}$.*
*The minimum possible dimension of $X^{\perp}$ is $\frac{n(n+1)}{2}$.*

# Some related theorems

Definiton For a linear subspace $X$ of $M_{m\times n}(\mathbb{F})$, define $X^\perp$ by

$$X^\perp = \{B \in M_{n\times m}(\mathbb{F}) : \mathrm{trace}(AB) = 0\ \forall\ A \in X\}.$$

Then $X^\perp$ is a linear space and $\dim(X) + \dim(X^\perp) = mn$.

**Theorem (Duality Theorem, Version 2)**

*Let $C \in \mathrm{GL}_n(\mathbb{F})$. Every element of the affine space $C + X$ is nonsingular (or has rank $n$) if and only if every one-dimensional subspace of $\mathbb{F}^n$ occurs as the rowspace of some element of $X^\perp \backslash X^\perp \cap C^\perp$.*
*The minimum possible dimension of $X^\perp$ is $\frac{n(n+1)}{2}$.*

# Some related theorems

Definiton For a linear subspace $X$ of $M_{m\times n}(\mathbb{F})$, define $X^{\perp}$ by

$$X^{\perp} = \{B \in M_{n\times m}(\mathbb{F}) : \text{trace}(AB) = 0 \ \forall \ A \in X\}.$$

Then $X^{\perp}$ is a linear space and $\dim(X) + \dim(X^{\perp}) = mn$.

**Theorem (Duality Theorem, Version 3)**

*Let $k \leq n$. Every element of the affine space $I_n + X$ has rank at least $k$ if and only if no element of $X$ has a non-zero eigenvalue in $\mathbb{F}$ whose geometric multiplicity exceeds $n-k$;*
*if and only if every $(n-k+1)$-dimensional subspace of $\mathbb{F}^n$ contains the rowspace of some element of $X^{\perp}$ of non-zero trace.*
*The minimum possible dimension of such an $X^{\perp}$ is $\frac{k(k+1)}{2}$.*

# Some related theorems

Definiton For a linear subspace $X$ of $M_{m\times n}(\mathbb{F})$, define $X^\perp$ by

$$X^\perp = \{B \in M_{n\times m}(\mathbb{F}) : \text{trace}(AB) = 0 \ \forall\ A \in X\}.$$

Then $X^\perp$ is a linear space and $\dim(X) + \dim(X^\perp) = mn$.

**Theorem (Duality Theorem, Version 4)**

*Let $C \in M_n(\mathbb{F})$ and let $k \leq n$. Every element of the affine space $C + X$ has rank at least $k$ if and only if every $(n-k+1)$-dimensional subspace of $\mathbb{F}^n$ contains the rowspace of some element of $X^\perp \backslash X^\perp \cap C^\perp$.*
*The minimum possible dimension of such an $X^\perp$ is $\frac{k(k+1)}{2}$.*

# Some related theorems

Definiton For a linear subspace $X$ of $M_{m\times n}(\mathbb{F})$, define $X^\perp$ by

$$X^\perp = \{B \in M_{n\times m}(\mathbb{F}) : \mathrm{trace}(AB) = 0 \ \forall \ A \in X\}.$$

Then $X^\perp$ is a linear space and $\dim(X) + \dim(X^\perp) = mn$.

**Theorem (Duality Theorem, Version 5)**

*Let $X$ be a subspace of $M_{m\times n}(\mathbb{F})$ and let $C \in M_{m\times n}(\mathbb{F})$. Let $k \leq \min(m,n)$. Then every element of the affine space $C + X$ has rank at least $k$ if and only if every subspace of dimension $m-k+1$ of $\mathbb{F}^m$ contains the rowspace of some element of $X^\perp \backslash X^\perp \cap C^\perp$. The minimum possible dimension of such an $X^\perp$ is $\frac{k(k+1)}{2}$.*

# Thank You

And thanks to the organisers!