# Partial matrices of constant rank over finite fields

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Providence

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# The setup

A partial matrix over a field $\mathbb{F}$ is a matrix whose entries are either specified entries of $\mathbb{F}$ or independent indeterminates. A completion of a partial matrix is a matrix obtained by assigning a value from $\mathbb{F}$ to each indeterminate entry.

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## Problems about Rank

- Given a partial matrix, what is the range of ranks of its completions?
- Characterize (all, or extremal examples of) partial matrices whose completions satisfy specified rank bounds, e.g. have constant rank.

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**Theorem (adapted from Huang and Zhan (2011))**

*Let $A$ be a $m \times n$ partial matrix of constant rank $r$ over a field $\mathbb{F}$. If $|\mathbb{F}| \geq \max(m, n)$ then $A$ possesses a $r \times r$ sub(partial)matrix whose completions all have rank $r$.*

# An Example

The following $3 \times 4$ partial matrix over $\mathbb{F}_2$ has all completions of rank 3, but possesses no $3 \times 3$ submatrix of constant rank 3.

$$\begin{pmatrix} 1 & X & 0 & 1 \\ 1 & 1 & Y & 0 \\ 1 & 0 & 1 & Z \end{pmatrix}$$

So *some* condition on the field order is necessary for the theorem to hold.

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# Some Observations

## Question

*$A$ is a $m \times n$ partial matrix of constant rank $r$ over a field $\mathbb{F}$, with $m \leq n$. If $A$ is exceptional (i.e. has no $r \times r$ submatrix of constant rank $r$), what can be said about $\mathbb{F}$, $m$ and $n$?*

- $A$ possesses constant columns (assumed linearly independent).
- Let $C$ be the subspace of $\mathbb{F}^m$ spanned by the constant columns. Then $1 \leq \dim C \leq r - 2$ and every element of $C^\perp$ includes at least one zero entry.
- If $|\mathbb{F}| \geq r$, then $\dim C \leq |\mathbb{F}| - 2$, and $C$ includes an element with exactly one non-zero entry. An induction argument produces an $r \times r$ submatrix of $A$ of constant rank $r$.

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# Exceptional cases occur only if $|\mathbb{F}| < r$

The following theorem can be proved by induction on $r$.

**Theorem**

*There exist exceptional $m \times n$ (with $m \leq n$) partial matrices of constant rank $r$ over $\mathbb{F}_q$ if and only if $r > q$ and $n \geq r + q - 1$.*

The base case: $r = q + 1$, $n \geq 2q$

An example with $q = 3:$ $(q + 1) \times (2q)$, exceptional of constant rank 4.

$$\begin{pmatrix} 1 & 1 & X & 1 & 1 & 1 \\ 1 & 2 & 1 & Y & 1 & 1 \\ 2 & 0 & 1 & 1 & Z & 1 \\ 0 & 2 & 2 & 1 & 1 & W \end{pmatrix}$$

# The case $r = q + 1$: need at least $2q$ columns

Let $A$ be a partial $m \times n$ matrix over $\mathbb{F}_q$ ($m \leq n$) of constant rank $q + 1$, and let $C \subset \mathbb{F}_q^m$ be the span of the constant columns of $A$.

- If $\dim C \geq q$, then $A$ is not exceptional.
- If $C$ contains an element with exactly one non-zero entry, then $A$ has a $(m-1) \times (n-1)$ submatrix of constant rank $q$, and $A$ is not exceptional.
- Otherwise $C^\perp$ has the "distributed zero property": every element of $C^\perp$ has at least one zero entry, but there is no position that is always zero in $C^\perp$.
- This means $\dim C \geq q - 1$, so if $A$ is exceptional, $\dim C = q - 1$ and $A$ has (exactly) $q - 1$ constant columns.

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# The case $r = q + 1$: at least $q + 1$ indeterminate columns

$A$ is a partial $m \times n$ matrix over $\mathbb{F}_q$ ($m \leq n$) of constant rank $q + 1$, and $C^\perp$ has the distributed zero property.

- Form $A'$ by assigning a value to all but one indeterminate in each indeterminate column of $A$.
- Given any $q$ positions in $\mathbb{F}_q^m$, there is an element $v$ of $C^\perp$ that has non-zero entries in all of them (this is because a vector space over $\mathbb{F}_q$ cannot be the union of $q$ hyperplanes).
- The indeterminates of $A'$ must collectively occupy at least $q + 1$ rows, otherwise $A'$ would have completions of different ranks.
- So $A'$ has at least $q + 1$ indeterminate columns, hence at least $2q$ columns in all.

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# The End

THANK YOU!

Advertisement If you are interested in this, see the talk by James McTigue on Thursday.