Affine spaces of matrices with lower rank bounds, and a dual property

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Plan

A property of matrix spaces

A dual property

- Trace bilinear form
- Dual Property
- Proof Summary

Generalizations and Extensions

- Partial Matrices

A Dimension Bound

Theorem (Quinlan 2011, de Seguins Pazzis 2011)

For any field $\mathbb{F}$ and any $n \geq 1$, if S is a linear subspace of $M_n(\mathbb{F})$ with the property that no element of S has a non-zero eigenvalue in $\mathbb{F}$, then

$$\dim S \leq \frac{n(n-1)}{2}.$$

Examples

1. $S = SUT_n(\mathbb{F})$, the space of strictly upper triangular matrices, for any field $\mathbb{F}$.
2. $S = A_n(\mathbb{R})$, the space of skew-symmetric $n \times n$ matrices over $\mathbb{R}$.

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A Symmetric Bilinear Form on $M_n(\mathbb{F})$

Define $\tau : M_n(\mathbb{F}) \times M_n(\mathbb{F}) \longrightarrow \mathbb{F}$ by

$$\tau(A, B) = \text{trace}(AB), \text{ for } A, B \in M_n(\mathbb{F}).$$

τ is a non–degenerate symmetric bilinear form on $M_n(\mathbb{F})$.

For a subspace W of $M_n(\mathbb{F})$, define the orthogonal complement of W by

$$W^\perp = \{X \in M_n(\mathbb{F}) : \text{trace}(XY) = 0 \ \forall \ Y \in W\}.$$

Then $W^\perp$ is a subspace of $M_n(\mathbb{F})$ and

- $\dim W + \dim W^\perp = n^2$
- $W_1 \subseteq W_2 \iff W_1^\perp \supseteq W_2^\perp$

A Dual Property

Notation

- T : the space of matrices of trace zero in $M_n(\mathbb{F})$
- $\mathbb{F}^n$: the space of row vectors with n entries in $\mathbb{F}$.
  A column vector is the transpose of an element of $\mathbb{F}^n$.

Theorem (Quinlan, 2011)

Let S be a subspace of $M_n(\mathbb{F})$. The following are equivalent

- *No element of S has a non-zero eigenvalue in $\mathbb{F}$*
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Back to the Examples

No element of S has a non-zero eigenvalue in $\mathbb{F}$	$\iff$	Every non-zero vector in $\mathbb{F}^n$ spans the rowspace of some element of rank 1 of $S^\perp \backslash S^\perp \cap T$

- If $S = SUT_n(\mathbb{F})$, then $S^\perp = UT_n(\mathbb{F})$, the space of *all* upper triangular matrices in $M_n(\mathbb{F})$.
  Every non-zero $v \in \mathbb{F}^n$ occurs as the only non-zero row of an upper triangular matrix of non-zero trace.
- If $S = A_n(\mathbb{R})$, then $S^\perp = S_n(\mathbb{R})$, the space of *symmetric* matrices in $M_n(\mathbb{F})$.
  If $v \in \mathbb{F}^n$, $v \neq 0$, then $v^T v$ is an element of $S_n(\mathbb{R})$ that has rowspace $\langle v \rangle$ and has non-zero trace.

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Quick proof, Part 1

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Proof ($\Longleftarrow$)

- Suppose $\lambda \in \mathbb{F}$ is an eigenvalue of some $X \in S$. We want to show $\lambda = 0$.
- Then $vX = \lambda v$ for some non-zero $v \in \mathbb{F}^n$.
- Let M_v be an element of $S^\perp \backslash S^\perp \cap T$ with rowspace $\langle v \rangle$.
- Then

$$M_v X = \lambda M_v \Longrightarrow \text{trace}(M_v X) = \lambda\, \text{trace}(M_v).$$

Since $\text{trace}(M_v X) = 0$ (as $M_v \in S^\perp$), and $\text{trace}(M_v) \neq 0$ (as $M_v \notin T$), we must have $\lambda = 0$.

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- Let $v \in \mathbb{F}^n$, $v \neq 0$. We want to show that some element of $S^\perp$ of non-zero trace has rowspace $\langle v \rangle$.
- The subspace $\{vX : X \in S\}$ of $\mathbb{F}^n$ does not contain v.
- So $\exists u \in \mathbb{F}^n$ with $(vX)u^T = 0 \ \forall \ X \in S$ and $vu^T \neq 0$.
- But $(vX)u^T = \text{trace}\left(u^T(vX)\right) = \text{trace}\left((u^T v)X\right)$ and $vu^T = \text{trace}(u^T v)$.
- So $u^T v$ has rowspace $\langle v \rangle$, has non-zero trace, and $\text{trace}(u^T v\, X) = 0 \ \forall X \in S \Longrightarrow u^T v \in S^\perp$.
- So $u^T v$ is the thing we want!

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Other versions of the duality theorem

$S^{\perp} \backslash S^{\perp} \cap T$ *"all 1-d rowspaces"* $\Longleftrightarrow$ S *"no non-zero eigenvalues"*

A subspace S of $M_n(\mathbb{F})$ has the property that no element possesses a non-zero eigenvalue in $\mathbb{F}$ if and only if every element of the affine subspace $I_n + S$ is non-singular.

Theorem (Duality Theorem, Version 1)

Every element of the affine space $I_n + S$ is non-singular if and only if every non-zero vector in $\mathbb{F}^n$ occurs as the rowspace of some element of $S^{\perp} \backslash S^{\perp} \cap T$.

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Theorem (Duality Theorem, Version 2)

Let $C \in \mathrm{GL}_n(\mathbb{F})$. Every element of the affine space $C + S$ is non-singular (or has rank n) if and only if every one-dimensional subspace of $\mathbb{F}^n$ occurs as the rowspace of some element of $S^{\perp} \backslash S^{\perp} \cap C^{\perp}$.

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A subspace S of $M_n(\mathbb{F})$ has the property that no element possesses a non-zero eigenvalue in $\mathbb{F}$ if and only if every element of the affine subspace $I_n + S$ is non-singular.

Theorem (Duality Theorem, Version 3)

Let $C \in M_n(\mathbb{F})$ and let $k \leq n$. Every element of the affine space $C + S$ has has rank at least k if and only if every $(n-k+1)$-dimensional subspace of $\mathbb{F}^n$ contains the rowspace of some element of $S^\perp \backslash S^\perp \cap C^\perp$.

Other versions of the duality theorem

$S^{\perp} \backslash S^{\perp} \cap T$ *"all 1-d rowspaces"* $\Longleftrightarrow$ S *"no non-zero eigenvalues"*

If S is a subspace of $M_{m\times n}(\mathbb{F})$, we define

$$S^{\perp} = \{X \in M_{n\times m}(\mathbb{F}) : \text{trace}(XY) = 0 \ \forall Y \in S\}.$$

Theorem (Duality Theorem, Version 4)

Let S be a subspace of $M_{m\times n}(\mathbb{F})$ and let $C \in M_{m\times n}(\mathbb{F})$. Let $k \leq \min(m, n)$. Then every element of the affine space $C + S$ has rank at least k if and only if every subspace of dimension $m - k + 1$ of $\mathbb{F}^m$ contains the rowspace of some element of $S^{\perp} \backslash S^{\perp} \cap C^{\perp}$.

Example

All ranks in the affine space $C + S \subseteq M_{m\times n}(\mathbb{F})$ are at least $k \iff$ every subspace of dimension $m-k+1$ of $\mathbb{F}^m$ contains the rowspace of some element of $S^\perp \backslash S^\perp \cap C^\perp$

$m = 4,\ n = 5,\ C = \mathrm{diag}(1,1,1),\ k = 3,\ m-k+1 = 2$
$S = \{A : A_{ij} = 0 \text{ for } j \leq i \leq 3\}$.

S

$$\begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & * & * \end{pmatrix}$$

$S^\perp$

$$\begin{pmatrix} * & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$C + S$

$$\begin{pmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ * & * & * & * & * \end{pmatrix}$$

$S^\perp \backslash S^\perp \cap C^\perp$

$$\begin{pmatrix} a_1 & * & * & 0 \\ 0 & a_2 & * & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$a_1 + a_2 + a_3 \neq 0$

Dimension Bounds

Theorem

Suppose that $C + X$ *is an affine subspace of* $M_{m\times n}(\mathbb{F})$ *in which every element has rank at least* r*. Then*

$$\dim X \leq mn - \frac{r(r+1)}{2}.$$

Theorem

Suppose that Y *is a subspace of* $M_{n\times m}(\mathbb{F})$ *with the following property : for some* $C \in M_{m\times n}(\mathbb{F})$*, every subspace of* $\mathbb{F}^m$ *of dimension* $m - r + 1$ *contains the rowspace of some element of* $Y \backslash Y \cap C^{\perp}$*. Then*

$$\dim Y \geq \frac{r(r+1)}{2}.$$

Partial Matrices

A partial $m \times n$ matrix A (over $\mathbb{F}$) is a matrix in which some entries are specified elements of $\mathbb{F}$ and the rest are independent indeterminates.

A completion of A is the matrix resulting from an assignment of specific values in $\mathbb{F}$ to each indeterminate.

Theorem (Brualdi, Huang & Zhan, 2010)

Let $\mathbb{F}$ be a field with at least $n+1$ elements. A partial $n \times n$ matrix A over $\mathbb{F}$ whose completions all have rank n can have at most $\frac{n(n-1)}{2}$ indeterminates. If this bound is attained, there exist permutation matrices P and Q for which PAQ is upper triangular with

- *non-zero constants on the main diagonal*
- *independent indeterminates above the main diagonal*
- *zeroes below the main diagonal.*

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An extension of this theorem

Theorem (McTigue & Quinlan, 2011)

Let $\mathbb{F}$ be any field. Let r, m, n be positive integers with $r \leq \min(m, n)$. A partial $m \times n$ matrix A over $\mathbb{F}$ whose completions all have rank at least r can have at most $mn - \frac{r(r+1)}{2}$ indeterminates.

If this bound is attained, then there exist permutation matrices P and Q for which PAQ has the following form :

- *All entries outside the upper left $r \times r$ region are indeterminates*
- *In the upper left $r \times r$ region, the entries on the main diagonal are non-zero constants, the entries above the main diagonal are indeterminates, and the entries below the main diagonal are zeroes.*

A glance at a proof (for $r = m = n$)

Every element of $C + X$ has rank n	$\iff$	Every non-zero vector in $\mathbb{F}^n$ spans the rowspace of some element of rank 1 of $X^{\perp} \backslash X^{\perp} \cap C^{\perp}$

A : a partial $n \times n$ matrix whose completions all have rank n. Write $A = C + X$

- C : "constant part" of A
- X : "indeterminate part" $\sim$ a linear subspace of $M_n(\mathbb{F})$.

Lemma

Then $X^{\perp}$ is described by the partial matrix Y that has indeterminates where X^T has zeroes, and zeroes where X^T has indeterminates.

As many indeterminates as possible in A (or X) means as few as possible in Y.

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Every element of $C + X$ has rank n	$\Longleftrightarrow$	Every non-zero vector in $\mathbb{F}^n$ spans the rowspace of some element of rank 1 of $X^\perp \backslash X^\perp \cap C^\perp$

This means

- Some row of Y has n indeterminates (say Row 1)
- Row 1 of $Y \sim \mathbb{F}^n$. Vectors that are orthogonal to Column 1 of $C^\perp$ form a hyperplane H.
- Every $v \in H$ must appear again in another row of Y.
- So Row 2 of Y must contain at least $n-1$ indeterminates (and this is enough only if Column 1 of C has only one non-zero entry).
- Y has at least $n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}$ indeterminates, and this bound is attained only if C is diagonal.

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- Row 1 of $Y \sim \mathbb{F}^n$. Vectors that are orthogonal to Column 1 of $C^\perp$ form a hyperplane H.
- Every $v \in H$ must appear again in another row of Y.
- So Row 2 of Y must contain at least $n - 1$ indeterminates (and this is enough only if Column 1 of C has only one non-zero entry).
- Y has at least $n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}$ indeterminates, and this bound is attained only if C is diagonal.

A glance at a proof (for $r = m = n$)

Every element of $C + X$ has rank n	$\iff$	Every non-zero vector in $\mathbb{F}^n$ spans the rowspace of some element of rank 1 of $X^\perp \backslash X^\perp \cap C^\perp$

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Thank You

- Danke schön!
- Go raibh maith agaibh!