Distinguishing covering groups of elementary abelian groups

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Automorphism groups

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Rank 1 covering groups

$p = 2$

Uniform covering groups

Groups and Graphs

Covering groups

Let A be an elementary abelian group of order p^n. A covering group of A is a group G with the following properties :

- $|G| = p^{n+\binom{n}{2}}$
- $G' = Z(G)$ is elementary abelian of order $p^{\binom{n}{2}}$
- $G/G' \cong A$

In this situation

- $G = \langle x_1, x_2, \ldots, x_n \rangle$
- $G' = \langle [x_i, x_j] \rangle_{i<j}$, elementary abelian of order $p^{\binom{n}{2}}$
- Each x_i^p is a unique product of the $[x_i, x_j]$.

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Examples and Construction

To designate a covering group $G = \langle x_1, \ldots, x_n \rangle$ of $A \cong C_p^n$, we specify a product of the $[x_i, x_j]$ to represent each x_i^p.

Example

$n = 2,\ p = 2,\ A \cong C_2 \times C_2$.
$G = \langle x_1, x_2 \rangle$, write $c = [x_1, x_2]$, then $G' = \langle c \rangle \cong C_2$. We have two choices each for x_1^2 and x_2^2 : 1 and c.

1. $x_1^2 = x_2^2 = 1$. Then G is non-abelian of order 8, generated by two involutions, $G \cong D_8$

So $C_2 \times C_2$ has two covering groups : three choices give D_8 and one gives Q_8.

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2. $x_1^2 = 1, x_2^2 = c$.

$$G = \langle x_1, x_2 | x_1^2 = 1, x_2^4 = 1, x_1 x_2 = x_2^3 x_1 \rangle.$$

Again $G \cong D_8$

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3. $x_1^2 = c, x_2^2 = 1$. This is the same as 2., again $G \cong D_8$.

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The essential automorphism group

If G is a covering group of $C_p^{(n)}$, then $\mathrm{Aut}\, G$ contains a subgroup $\mathrm{CAut}\, G$ of order $p^{n\binom{n}{2}}$, consisting of all elements that induce the identity on G/G'.

The essential automorphism group $\overline{\mathrm{Aut}}\, G$ of G is the group of automorphisms of G/G' that are induced by automorphisms of G. Thus

- $\overline{\mathrm{Aut}}\, G \cong \dfrac{\mathrm{Aut}\, G}{\mathrm{CAut}\, G}$
- $\overline{\mathrm{Aut}}\, G$ is a subgroup of $\mathrm{GL}(n, p)$.

Example

- $\overline{\mathrm{Aut}}\, D_8$ is cyclic of order 2.
- $\overline{\mathrm{Aut}}\, Q_8 \cong S_3 \cong \mathrm{GL}(2, 2)$.

The pth power map

If G is a covering group of $C_p^{(n)}$, define the pth power map $\phi : G \longrightarrow G'$ by

$$\phi(x) = x^p.$$

In general

$$(xy)^p = x^p y^p [y,x]^{p(p-1)/2} = \begin{cases} x^p y^p & \text{if } p \text{ is odd} \\ x^2 y^2 [x,y] & \text{if } p = 2 \end{cases}$$

So ϕ is a group homomorphism if (and only if) p is odd. In this case the set G^p of pth powers in G is a subgroup of G' of order dividing p^n.

Definition

If p is odd, G is said to have rank k if $|G^p| = p^k$.

Ranks 0 and 1 (p odd)

$C_p^{(n)}$ has a unique covering group of rank 0 (exponent p).

Question *How many of rank 1?*

Suppose G is a covering group of $C_p^{(n)}$ of rank 1.

Then G^p is a characteristic subgroup of G of order p, generated by some $r \in G'$. Let t be the least integer for which r is the product of t simple commutators in G'.

Note: Let $V = G/G'$, interpreted as a vector space over $\mathbb{F}_p$. Then $G' \cong V \wedge V$, via $[x, y] \leftrightarrow x \wedge y$.

Observation $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$. The number of isomorphism types of covering groups of rank 1 of $C_p^{(n)}$ is at least $\lfloor \frac{n}{2} \rfloor$.

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Classifying covering groups of rank 1 (odd p)

Suppose G is a covering group of rank 1 of $C_p^{(n)}$. Then exactly one of the following happens.

1. G has a basis $\{x_1, \ldots, x_n\}$ for which $x_i^p = 1$ for $i = 2, \ldots, n$ and

$$x_1^p = [x_1, x_2][x_3, x_4] \ldots [x_{2t-1}, x_{2t}]$$

for some t with $2 \leq 2t \leq n$, or

2. G has a basis $\{x_1, \ldots, x_n\}$ for which $x_i^p = 1$ for $i = 2, \ldots, n$ and

$$x_1^p = [x_2, x_3][x_3, x_4] \ldots [x_{2t}, x_{2t+1}]$$

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Theorem

The number of isomorphism types of covering groups of rank 1 of $C_p^{(n)}$ is $n - 1$.

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Uniform covering groups : $p = 2$

If G is a covering group of $C_2^{(n)}$, the squaring map in G satisifies

$$(xy)^2 = x^2y^2[x, y],$$

so it is not a homomorphism/linear transformation.

For odd p, a rank 1 covering group of $C_p^{(n)}$ is one possessing a set of n generators all of whose elements have the same (non-identity) pth power.

By analogy, we can consider covering groups of $C_2^{(n)}$ having n generators all having the same square. We refer to such examples as *uniform*.

Question *How many uniform covering groups does $C_2^{(n)}$ have?*

Graphs

Suppose G is a uniform covering group of C_2^n, with uniform basis $\{x_1, \dots, x_n\}$, $x_i^2 = r \in G'$. Then

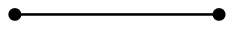
$$r = \prod_{1 \le i < j \le n} [x_i, x_j]^{a_{ij}} \ : \ a_{ij} = 0 \text{ or } a_{ij} = 1$$

We can represent this with a graph on vertex set $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$, and with $\bar{x}_i$ and $\bar{x}_j$ adjacent whenever $a_{ij} = 1$.

Examples

$G = D_8 = \langle x_1, x_2 \rangle$
$r = x_1^2 = x_2^2 = 1$

$G = Q_8 = \langle x_1, x_2 \rangle$
$r = x_1^2 = x_2^2 = [x_1, x_2]$



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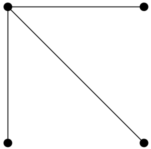
Examples

$n = 4$

$r = [x_1, x_2][x_1, x_3][x_1, x_4]$



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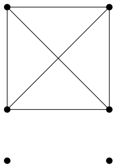
Examples

$n = 6$

$r = [x_1, x_2][x_1, x_3][x_1, x_4][x_2, x_3][x_2, x_4][x_3, x_4]$



Classifying uniform covering groups

Suppose $n \geq 3$ and $G = \langle x_1, x_2, \ldots, x_n \rangle$ is a uniform covering group of $C_2^{(n)}$ with $x_i^2 = r$ for $i = 1, \ldots, n$.

Lemma

1. *If $s \in G'$, $s \neq r$, then at most two cosets of G' in G can consist of elements with square s.*
2. *If $x \in G$ and $x^2 = r$, then either*
   - $x \in x_i G'$ *for some* i
   - $x \in G'$ *and* $r = 1$, *or*
   - $x \in x_{i_1} x_{i_2} \ldots x_{i_{2k}} G'$ *and*

$$r = \prod_{1 \leq l < m \leq 2k} [x_{i_l}, x_{i_m}]$$

for some subset $\{i_1, i_2, \ldots, i_{2k}\}$ *of* $\{1, \ldots, n\}$

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Suppose $n \geq 3$ and $G = \langle x_1, x_2, \ldots, x_n \rangle$ is a uniform covering group of $C_2^{(n)}$ with $x_i^2 = r$ for $i = 1, \ldots, n$.

Lemma

1. *There is only one candidate for r in G'.*
2. *There is only one candidate for a set of n cosets of G' in G from which the elements of a uniform basis can be drawn, except in the special case where the graph describing r consists of a complete graph on a positive even number $2k$ of vertices, and $n - 2k$ isolated vertices. In this case $n + 1$ cosets (other than G') consist of elements with square r. All uniform bases determine the same graph.*

Classifying uniform covering groups

Theorem

- *A uniform covering group G of $C_2^{(n)}$ is determined by its graph $\Gamma(G)$.*
- *The number of uniform covering groups of $C_2^{(n)}$ is the number of graphs on n vertices.*
- $\overline{\text{Aut}}\, G \cong \text{Aut}\, \Gamma(G)$, *except* *when* $\Gamma(G) \cong K_{2k} \cup N_{n-2k}$ *and* $k \geq 1$, *then*

$$\overline{\text{Aut}}\, G \cong S_{2k+1} \times S_{n-2k}.$$

Special Case: If n is even and $\Gamma(G)$ is the complete graph on n vertices, then $\overline{\text{Aut}}\, G \cong S_{n+1}$.

Thank You

Happy 100th Birthday to Mathematics at the University of Alberta!