What I did on my sabbatical

Rachel Quinlan

School of Mathematics, Statistics and Applied Mathematics
rachel.quinlan@nuigalway.ie

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Plan

A Dimension Bound

Theorem (Quinlan 2011, de Seguins Pazzis 2011)

For any field $\mathbb{F}$ and any $n \geq 1$, if S is a linear subspace of $M_n(\mathbb{F})$ with the property that no element of S has a non-zero eigenvalue in $\mathbb{F}$, then

$$\dim S \leq \frac{n(n-1)}{2}.$$

Examples

1. $S = SUT_n(\mathbb{F})$, the space of strictly upper triangular matrices, for any field $\mathbb{F}$.
2. $S = A_n(\mathbb{R})$, the space of skew-symmetric $n \times n$ matrices over $\mathbb{R}$.

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Remarks on the Examples

1. Theorem The maximum possible dimension of a subspace of $M_n(\mathbb{F})$ in which every element is nilpotent is $\frac{n(n-1)}{2}$. Every subspace of $M_n(\mathbb{F})$ that consists of nilpotent elements *and* has dimension $\frac{n(n-1)}{2}$ is similar to $SUT_n(\mathbb{F})$.
   (Gerstenhaber 1958)
2. It is not true that *every* space of nilpotent matrices is triangularizable.
3. A space of nilpotent matrices that has the additional structure of a Lie Algebra is always triangularizable. Example 2 shows that this is not true if "nilpotent" is relaxed to the condition of Theorem 1.

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A Symmetric Bilinear Form on $M_n(\mathbb{F})$

Define $\tau : M_n(\mathbb{F}) \times M_n(\mathbb{F}) \longrightarrow \mathbb{F}$ by

$$\tau(A, B) = \text{trace}(AB), \text{ for } A, B \in M_n(\mathbb{F}).$$

τ is a non–degenerate symmetric bilinear form on $M_n(\mathbb{F})$.
For a subspace W of $M_n(\mathbb{F})$, define the orthogonal complement of W by

$$W^{\perp} = \{X \in M_n(\mathbb{F}) : \text{trace}(XY) = 0 \ \forall \ Y \in W\}.$$

Then $W^{\perp}$ is a subspace of $M_n(\mathbb{F})$ and

- $\dim W + \dim W^{\perp} = n^2$
- $W_1 \subseteq W_2 \iff W_1^{\perp} \supseteq W_2^{\perp}$
- $\langle I_{n\times n} \rangle^{\perp} = T$ (the kernel of the trace mapping).

A Dual Property

Notation

- T : the space of matrices of trace zero in $M_n(\mathbb{F})$
- $\mathbb{F}^n$: the space of row vectors with n entries in $\mathbb{F}$. A column vector is the transpose of an element of $\mathbb{F}^n$.

Theorem

Let S be a subspace of $M_n(\mathbb{F})$. Then no element of S has a non-zero eigenvalue in $\mathbb{F}$ if and only if every one-dimensional subspace of $\mathbb{F}^n$ occurs as the rowspace of some element of $S^\perp \backslash S^\perp \cap T$.

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No element of S has a non-zero eigenvalue in $\mathbb{F}$	$\iff$	Every non–zero vector in $\mathbb{F}^n$ spans the rowspace of some element of rank 1 of $S^\perp \backslash S^\perp \cap T$

- If $S = SUT_n(\mathbb{F})$, then $S^\perp = UT_n(\mathbb{F})$, the space of *all* upper triangular matrices in $M_n(\mathbb{F})$.
  Every non–zero $v \in \mathbb{F}^n$ occurs as the only non–zero row of an upper triangular matrix of non–zero trace.
- If $S = A_n(\mathbb{R})$, then $S^\perp = S_n(\mathbb{R})$, the space of *symmetric* matrices in $M_n(\mathbb{F})$.
  If $v \in \mathbb{F}^n$, $v \neq 0$, then $v^T v$ is an element of $S_n(\mathbb{R})$ that has rowspace $\langle v \rangle$ and has non–zero trace.

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Quick proof, Part 1

No element of S has a non-zero eigenvalue in $\mathbb{F}$	$\iff$	Every non–zero vector in $\mathbb{F}^n$ spans the rowspace of some element of rank 1 of $S^\perp \backslash S^\perp \cap T$

Proof ($\Longleftarrow$)

- Suppose $\lambda \in \mathbb{F}$ is an eigenvalue of some $X \in S$. We want to show $\lambda = 0$.
- Then $vX = \lambda v$ for some non–zero $v \in \mathbb{F}^n$.
- Let M_v be an element of $S^\perp \backslash S^\perp \cap T$ with rowspace $\langle v \rangle$.
- Then

$$M_v X = \lambda M_v \Longrightarrow \text{trace}(M_v X) = \lambda \, \text{trace}(M_v).$$

Since $\text{trace}(M_v X) = 0$ (as $M_v \in S^\perp$), and $\text{trace}(M_v) \neq 0$ (as $M_v \notin T$), we must have $\lambda = 0$.

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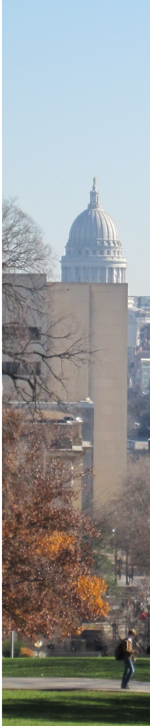
- Let $v \in \mathbb{F}^n$, $v \neq 0$. We want to show that some element of $S^\perp$ of non–zero trace has rowspace $\langle v \rangle$.
- The subspace $\{vX : X \in S\}$ of $\mathbb{F}^n$ does not contain v.
- So $\exists u \in \mathbb{F}^n$ with $(vX)u^T = 0 \; \forall \; X \in S$ and $vu^T \neq 0$.
- But $(vX)u^T = \text{trace}\left(u^T(vX)\right) = \text{trace}\left((u^T v)X\right)$ and $vu^T = \text{trace}(u^T v)$.
- So $u^T v$ has rowspace $\langle v \rangle$, has non–zero trace, and $\text{trace}(u^T v \, X) = 0 \; \forall X \in S \Longrightarrow u^T v \in S^\perp$.
- So $u^T v$ is the thing we want!



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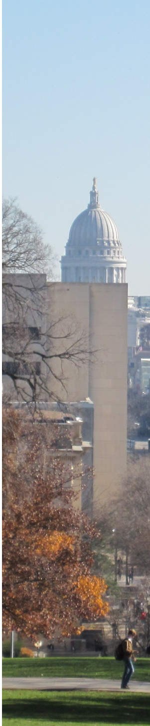

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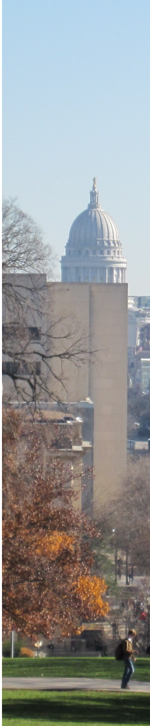

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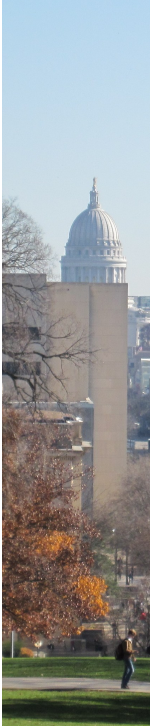

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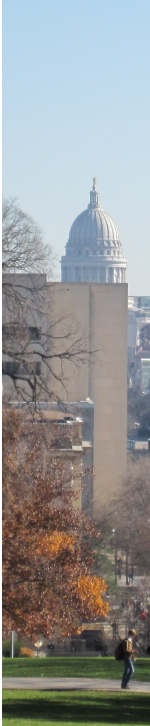

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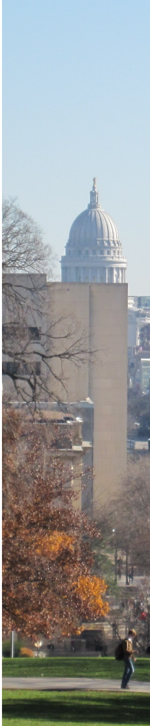

Two Consequences

$R\backslash R\cap T$ *"all 1-d rowspaces"* $\Longleftrightarrow$ $R^{\perp}$ *"no non–zero eigenvalues"*

1. D : maximum dimension of a subspace with *"no non-zero eigenvalues in* $\mathbb{F}$*"*
   d : minimum dimension of a subspace with *"all 1-d rowspaces on elements of non-zero trace"*

$$D + d = n^2.$$

2. For any subspace W of $M_n(\mathbb{F})$, $(W^{\perp})^T = (W^T)^{\perp}$. The "no non–zero eigenvalue" property is preserved under transposition $\Longrightarrow$ so is the dual property :

Every one-dimensional rowspace occurs in $R\backslash R\cap T$ $\Longleftrightarrow$ Every one-dimensional rowspace occurs in $R^T\backslash R^T\cap T$

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Another Dimension Bound

Theorem

Let R be a subspace of $M_n(\mathbb{F})$ with the property that every one-dimensional subspace of $\mathbb{F}^n$ is the rowspace of some element of R of non-zero trace. Then

$$\dim R \geq \frac{n(n+1)}{2}.$$

Note The property of R is preserved under similarity.

"Basis-free" version of the property and Theorem

Let V be a $\mathbb{F}$-vector space, $\dim V = n$. Let $\mathcal{R}$ be a subspace of $\mathrm{End}_{\mathbb{F}}(V)$ with the property that for every hyperplane H of V, some element of $\mathcal{R}$ of non-zero trace annihilates H. Then

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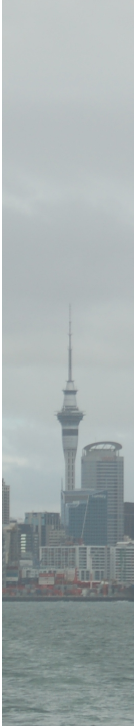

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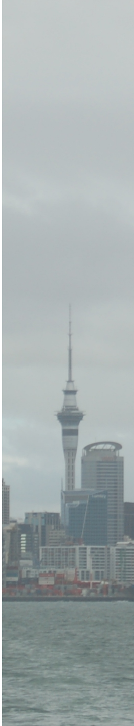

Induction Machinery

$\mathcal{R}$: a subspace of $\mathrm{End}_{\mathbb{F}}(V)$, with an element of non-zero trace annihilating H, for each hyperplane H of V.
R : a matrix realization of $\mathcal{R}$.

- R_0 : space of elements of R with zero 1st column.
- P : space of 1st columns of elements of R.
  Then $\dim R = \dim R_0 + \dim P$.
- R^* : the subspace of $M_{n-1}(\mathbb{F})$ obtained by deleting the first row and column from every element of R_0.
- By induction, $\dim R^* \geq \frac{n(n-1)}{2}$, so $\dim R_0 \geq \frac{n(n-1)}{2}$

Would like to show: there is a basis of V with respect to which every element of $(\mathbb{F}^n)^T$ occurs as Column 1 of an element of R.

Enough to do this for Row 1, by the "transposability".

This bit will be over soon – don't panic

1. $\exists\theta_1 \in \mathcal{R}$ with : $\text{trace}\,\theta_1 = 1,\ \text{rank}\,\theta_1 = 1$
   Choose b_1 to span $\text{Im}\,\theta_1$.
2. $\exists\theta_2 \in \mathcal{R}$ with :
   $\text{trace}\,\theta_2 = 1,\ \text{rank}\,\theta_2 = 1,\ \theta_2(b_1) = 0$
   Choose x_2 to span $\text{Im}\,\theta_2$.
   Note $x_2 \notin \langle b_1 \rangle$ as $\text{trace}\,\theta_2 \neq 0$
   Write $b_2 = x_2 - b_1$. Then $\theta_2(b_2) = b_2 + b_1$.
3. $\exists\theta_3 \in \mathcal{R}$ with:
   $\text{trace}\,\theta_3 = 1,\ \text{rank}\,\theta_3 = 1,\ \theta_3(b_1) = \theta_3(b_2) = 0$
   Choose x_3 to span $\text{Im}\,\theta_3$. Note $x_3 \notin \langle b_1, b_2 \rangle$.
   Write $b_3 = x_3 - b_1$. Then $\theta_3(b_3) = b_3 + b_1$.

Keep going . . . to get a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ of V.
Row 1 of the matrix of θ_i with respect to $\mathcal{B}$ starts with $i-1$ zeroes followed by a 1 in position i.
So the space of "first rows" is all of $\mathbb{F}^n$

This bit will be over soon – don't panic

1. $\exists \theta_1 \in \mathcal{R}$ with : $\operatorname{trace}\theta_1 = 1,\ \operatorname{rank}\theta_1 = 1$
   Choose b_1 to span $\operatorname{Im}\theta_1$.
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   Note $x_2 \notin \langle b_1 \rangle$ as $\operatorname{trace}\theta_2 \neq 0$
   Write $b_2 = x_2 - b_1$. Then $\theta_2(b_2) = b_2 + b_1$.
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Keep going . . . to get a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ of V.
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Other versions of the duality theorem

$S^\perp \backslash S^\perp \cap T$ *"all 1-d rowspaces"* $\Longleftrightarrow$ S *"no non-zero eigenvalues"*

A subspace S of $M_n(\mathbb{F})$ has the property that no element possesses a non-zero eigenvalue in $\mathbb{F}$ if and only if every element of the affine subspace $I_n + S$ is non-singular.

Theorem (Duality Theorem, Version 1)

Every element of the affine space $I_n + S$ is non-singular if and only if every non-zero vector in $\mathbb{F}^n$ occurs as the rowspace of some element of $S^\perp \backslash S^\perp \cap T$.

Other versions of the duality theorem

$$\boxed{S^\perp \backslash S^\perp \cap T \text{ “all 1-d rowspaces”} \iff S \text{ “no non-zero eigenvalues”}}$$

A subspace S of $M_n(\mathbb{F})$ has the property that no element possesses a non-zero eigenvalue in $\mathbb{F}$ if and only if every element of the affine subspace $I_n + S$ is non-singular.

Theorem (Duality Theorem, Version 2)

Let $C \in \mathrm{GL}_n(\mathbb{F})$. Every element of the affine space $C + S$ is non-singular (or has rank n) if and only if every one-dimensional subspace of $\mathbb{F}^n$ occurs as the rowspace of some element of $S^\perp \backslash S^\perp \cap C^\perp$.

Other versions of the duality theorem

$$\boxed{S^\perp \backslash S^\perp \cap T \text{ ``all 1-d rowspaces''} \iff S \text{ ``no non-zero eigenvalues''}}$$

A subspace S of $M_n(\mathbb{F})$ has the property that no element possesses a non-zero eigenvalue in $\mathbb{F}$ if and only if every element of the affine subspace $I_n + S$ is non-singular.

Theorem (Duality Theorem, Version 3)

Let $C \in M_n(\mathbb{F})$ and let $k \leq n$. Every element of the affine space $C + S$ has has rank at least k if and only if every $(n - k + 1)$-dimensional subspace of $\mathbb{F}^n$ contains the rowspace of some element of $S^\perp \backslash S^\perp \cap C^\perp$.

Other versions of the duality theorem

$S^{\perp} \backslash S^{\perp} \cap T$ *"all 1-d rowspaces"* $\iff$ S *"no non-zero eigenvalues"*

If S is a subspace of $M_{m \times n}(\mathbb{F})$, we define

$$S^{\perp} = \{X \in M_{n \times m}(\mathbb{F}) : \mathrm{trace}(XY) = 0 \; \forall Y \in S\}.$$

Theorem (Duality Theorem, Version 4)

Let S be a subspace of $M_{m \times n}(\mathbb{F})$ and let $C \in M_{m \times n}(\mathbb{F})$. Let $k \leq \min(m, n)$. Then every element of the affine space $C + S$ has rank at least k if and only if every subspace of dimension $m - k + 1$ of $\mathbb{F}^m$ contains the rowspace of some element of $S^{\perp} \backslash S^{\perp} \cap C^{\perp}$.

Background

Duality and Involutions in Representation Theory, August 2008

Question (*F. Szechtman, 2003, motivation in group theory*)
Suppose $\mathcal{R}_1$ is a subspace of $\mathrm{End}_{\mathbb{F}}(V)$ with the following property :
$\exists g \in \mathrm{End}_{\mathbb{F}}(V)$, $g \notin \mathcal{R}_1$, so that for every hyperplane H of V, some element of $\mathcal{R}_1$ agrees with g on H.
What is the minimum possible dimension of $\mathcal{R}_1$?

Reformulations of Szechtman's Question

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1. $g + \mathcal{R}_1$ is a (non-linear) *affine subspace* of $\mathrm{End}_{\mathbb{F}}(V)$ that contains for every hyperplane H of V an element annihilating H.
   Minimum dimension of such an affine subspace?
2. Write $\mathcal{R} = \langle g, \mathcal{R}_1 \rangle$. Then $\mathcal{R}_1$ has codimension 1 in $\mathcal{R}$.
   Minimum dimension of a subspace $\mathcal{R}$ of $\mathrm{End}_{\mathbb{F}}(V)$ for which $\mathcal{R} \backslash \mathcal{R}_1$ has elements annihilating all hyperplanes, for some $\mathcal{R}_1$ of codimension 1 in $\mathcal{R}$?
   We've answered this if $\mathcal{R}_1 = \mathcal{R} \cap$(trace kernel) - this restriction does not change the minimum dimension.

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Burnside, 1913

On the outer automorphisms of a group, Proc. LMS

> *"WHILE preparing the second edition of my Theory of Groups for the press I made many ineffectual attempts to determine whether an outer isomorphism of a group necessarily permutes some of its conjugate sets, or, in the alternative, if groups exist some outer isomorphisms of which change every operation into a conjugate operation. I have since succeeded in constructing comparatively simple examples showing that of the two suppositions the latter is the correct one. One of the simplest of these is given below."*

Neumann, 1981

Not quite inner automorphisms, Bull. Australian MS

> *"Burnside asked the question whether an automorphism that maps each element of a group onto a conjugate element must be an inner automorphism, and after "many ineffectual attempts" constructed examples to answer his question in the negative . . . Gerhard Kowol has asked (oral communication) whether there is a group G with an outer automorphism that coincides on each triplet of elements with an inner automorphism (depending, of course, on the triplet), but that does not agree with any inner automorphism on some quadruplet of elements . . . I here present examples for all finite* $n \geq 2$*."*

n–Inner automorphisms of finite groups

F. Szechtman, AMS Proceedings, Vol. 131, 2003

An automorphism θ of a finite group G is *n-inner* if on every subset of G with fewer than n elements, it coincides with some inner automorphism.

Szechtman defined an action of the additive group of $M_n(\mathbb{F}_p)$ on $\mathbb{F}_p^n \oplus \mathbb{F}_p^n$; hence a semidirect product

$$\Gamma = (\mathbb{F}_p^n \oplus \mathbb{F}_p^n) \rtimes E$$

for any subgroup E of $M_n(\mathbb{F}_p)$.

Every element σ of $M_n(\mathbb{F}_p)$ defines an automorphism of Γ

-this is *outer* if σ does not belong to E.

-it is n-inner if σ agrees with some element of E on every subspace of dimension less than n.

Thank You !!