A proof evaluation exercise in an introductory linear algebra course

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The Context

- The course was an introduction to linear algebra in Semester 1 for approximately 100 first year students taking specialist mathematics.
- Homework 1 included a proof evaluation exercise in which students were asked to comment on five purported proofs of a statement about linear transformations.
- Several reasons for including this exercise
  - hope of prompting engagement with the purposes and mechanisms of mathematical proof;
  - hope of learning something about students' thinking;
  - hope that it might help students to develop a sense of personal responsibility for assessing the correctness of the mathematics that they read.

The Task

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Alison, Bob, Charlie, Deirdre and Ed are thinking about proving the following statement.

If the function $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation, then T fixes the origin, i.e. $T(0,0) = (0,0)$.

Students were presented with the above text and with five "proofs" proposed by Alison, Bob, Charlie, Deirdre and Ed. They were asked to

- Decide whether each proposed proof is correct or not (with explanation if not).
- Rank them in order of preference, with explanation.

Alison's Answer

Suppose that $T(1,1) = (a,b)$. Then

$$T[(1,1)+(0,0)] = T(1+0,1+0) = T(1,1) = (a,b).$$

But on the other hand since T respects addition

$$\begin{aligned} T[(1,1)+(0,0)] &= T(1,1) + T(0,0) \\ &= (a,b) + T(0,0) = (a,b) \text{ from above.} \end{aligned}$$

So $T(0,0) = (a,b) - (a,b) = (0,0)$.

Responses to Alison's proof - *showing is telling*

28 responses, 17 approvals, 10 rejections, 1 non-commital

For mathematicians, show and prove are more or less synonymous. In particular, show has a special meaning. Possibly not for students.

Students variously asserted that

- Alison uses the fact that T respects addition . . .
- Alison shows that T respects addition . . .

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The following comment may contain both senses of show:

She proves it by *showing* that points of linear transformation are additive and she manipulated this to *show* that the origin will be fixed.

If we interpret show in the everyday sense of point out or draw attention to, this is completely reasonable.

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Two students seemed to realize in the course of this that the use of show is problematic.

- Alison's proof ~~shows~~ uses the fact that linear transformations respect addition . . .
- She ~~shows~~ knows that it respects addition because it is a linear transformation.

Bob's Answer

We know that for any element u of $\mathbb{R}^2$ and for any real number k we have

$$T(ku) = kT(u).$$

Then applying T to $(0,0)$ and multiplying the result by any real number k must give the same result as multiplying $(0,0)$ by k first and then applying T. But multiplying $(0,0)$ by k always results in $(0,0)$ no matter what the value of k is. So it must be that the image under T of $(0,0)$ is a point in $\mathbb{R}^2$ which does not change when it is multiplied by a scalar. The only such point is $(0,0)$. So it must be that $T(0,0) = (0,0)$.

Responses to Bob's proof - *half the story*

28 responses, 21 approvals, 6 rejections, 1 partial approval

Two reasons for rejection.

1. *2 students* objected to the assertion that

   $(0,0)$ is the only element of $\mathbb{R}^2$ that does not change when multiplied by a scalar

   These students pointed out that (for example) $(2,3)$ does not change when it is multiplied by the scalar 1.

2. *4 students who rejected Bob's proof and 3 who accepted it* complained that it uses only part of the definition of a linear transformation. All seven criticized Alison's proof on the same grounds.

   Bob supplies the other half of Alison's proof, he proves the statement for scalar multiplication. He is also half right.

Considering the performance of proof evaluation

Pfeiffer (2010) proposed a theoretical tool for discussion of what a person does when evaluating a mathematical proof.

- Proofs (and proof) are regarded as artefacts of mathematical practice.
- An artefact is intentionally made by an author (or authors) for a definite purpose. Its existence and character depend on the author's productive intention.
- The quality of an artefact can only be judged on the basis of its effectiveness at achieving its purpose. Evaluation of an artefact might involve consideration of
  - The suitability of the author's intention for the purpose;
  - The match of the author's intention to the actual character of the artefact;
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Back to the objections to Bob's proof

1. $(0, 0)$ is the only element of $\mathbb{R}^2$ that does not change when multiplied by a scalar

   Maybe a legitimate criticism of a mismatch between the intended and actual characters of Bob's proof, but is this a legitimate reason to reject it?

2. Bob's and Alison's proofs each use only part of the definition of a linear transformation.

   This objection seems to be based on superficial inspection of the "ingredients" of these proofs rather than on careful consideration of the appropriateness of Bob's approach for his purpose.

Remark

No student commented that both Alison and Bob prove a more general statement than the one mentioned.

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Charlie's Answer

Think of T as the function that moves every point one unit to the right. So T moves the point $(0, 0)$ to the point $(1, 0)$. Then T is a linear transformation but T does not fix the origin.

This example proves that the statement is not true.

Responses to Charlie's proof - *true and false*

25 of 28 students considered that Charlie's answer does not prove that the statement is not true.

Grounds for Objection

- 11 objected on the grounds that Charlie's function is not (or may not be) a linear transformation.
- 2 on the grounds that the statement is true.
- 6 on the grounds that Charlie only considers one example. No sign in the work of these students of consideration of Charlie's (exceptional) intention and its connections to his purpose and to his argument.

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Responses to Charlie's Proof - *true and false*

3 students approved Charlie's proof. Each of these three also accepted at least one of the other proofs and appeared to indicate a belief that the statement could be both validated by a correct proof and contradicted by a counterexample. One commented

> *As he notes, the linear transformation could possibly move every point one unit to the right. Therefore T does not fix the origin.*

and (on ranking Charlie's proof 3rd)

> *Even though Charlie disproves the statement, it's a very valid reason to disprove it.*

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Deirdre's Answer

Suppose that (a, b) is a point in $\mathbb{R}^2$ for which $T(a, b) = (0, 0)$. Then

$$T(2a, 2b) = T[2(a, b)] = 2T(a, b) = (0, 0).$$

Thus $T(2a, 2b) = T(a, b)$, so $(2a, 2b) = (a, b)$, so $2a = a$ and $2b = b$. Thus $a = 0$, $b = 0$ and $T(0, 0) = (0, 0)$.

Responses to Deirdre's proof - *probably ok*

19 students accepted Deirdre's proof as correct. 8 rejected it.

(My) Comments on Deirdre's argument

- Deirdre attempts to prove that if $T(a, b) = (0, 0)$ for a linear transformation T, then $(a, b) = (0, 0)$. Since this statement is not true there is no hope of proving it. Even if it were true, the desired conclusion would not be an automatic consequence.
- Deirdre's proof contains the erroneous deduction that

$$T(2a, 2b) = T(a, b) \Longrightarrow (2a, 2b) = (a, b).$$

Responses to Deirdre's proof - *probably ok*

19 students accepted Deirdre's proof as correct. 8 rejected it.

Students accepting Deirdre's argument

- Many students who accepted Deirdre's proof identified similarities with Bob's. It is possible that this compromised their vigilance in assessing Deirdre's argument.
- At least three students accepted Deirdre's proof despite writing that they did not understand it - this behaviour contrasts with that of experienced mathematicians.

The way Deirdre's answer is laid out is confusing. I think that it probably does prove it but she needs to write it better.

Responses to Deirdre's proof - *probably ok*

19 students accepted Deirdre's proof as correct. 8 rejected it.

Students rejecting Deirdre's argument

- Two students (and two more who accepted it) criticized the use of the scalar 2 instead of a general k.
- Two rejected the proof on the basis of the "internal" error.

Her second line contains a mistake, when she states that $T(2a, 2b) = T(a, b) \implies (2a, 2b) = (a, b)$. This is not necessarily true. She is incorrect.

Remark

Not one student noted the serious flaw in the logical structure of Deirdre's argument.

Ed's Answer

Since T is a linear transformation it can be represented by a matrix. Suppose that the matrix of T is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Then the image of $(0, 0)$ under T can be calculated as follows.

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a0 + c0 \\ b0 + d0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $T(0, 0) = (0, 0)$.

Responses to Ed's proof - *clear and simple*

All 28 students accepted Ed's proof as correct, and 18 ranked it 1st.

- Few students commented in detail - typcial remarks said it was clear, simple, short, easy to understand . . .
- Overall the group indicated a clear preference for Ed's translation of the problem into an easy matrix calculation over Alison and Bob's reasoning from the defining properties of a linear transformation.
- A more experienced mathematician might feel that Ed's proof establishes the truth of the statement without offering much insight into why it is true.

Concluding remarks

- Proof evaluation constitutes a large part of our practice as mathematicians, but (in my experience at least) receives very little explicit attention in curricula.
- It appears not to be automatic that students develop the mathematician's habit of critical and sceptical reading of proofs. If this is a desired outcome, asking them to validate and evaluate (sometimes flawed) proofs may be helpful.
- Such tasks may also have the advantage of allowing novice students a real opportunity to assume and exercise personal mathematical authority.

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