Proof. We count, in two ways, the number of ordered triples of the form (u, v, w) in G with the property that v is adjacent to both u and w and that u and w are not adjacent to each other.

Suppose we choose v first - we have n choices here. Regardless of how this choice is made, the number of choices avaiable for a neighbour u of v is k. Having chosen u, the final step is to choose a vertex w that is adjacent to v but is not a common neighbour of u and v. There are $k - 1$ neighbours of v from which w may be chosen, but λ of these are also neighbours of u. So the number of choices for w is $k - 1 - \lambda$. Hence the number of choices for the triple (u, v, w) is $nk(k - \lambda - 1)$.

On the other hand suppose we choose u first. We have n choices for u, and then we may choose w from among the $n - k - 1$ non-neighbours of u. Having done this we have μ choices for v among the common neighbours of u and w. So the number of choices for the triple (u, v, w) is $n(n - k - 1)\mu$.

Putting these two counts together we find

$$nk(k - \lambda - 1) = n(n - k - 1)\mu \implies k(k - \lambda - 1) = (n - k - 1)\mu.$$

□

4.2 The Adjacency Spectrum of a strongly regular graph

In this section we use the defining properties of a strongly regular graph to show that the adjacency matrix of such a graph satisfies a particular quadratic equation, from which we deduce that the adjacency spectrum can have at most three distinct elements. It is clear that the adjacency matrix of a k-regular graph has k as an eigenvalue with corresponding eigenvector $\mathbf{1}$, since the row sums are all equal to k.

Lemma 4.2.1. *Let G be a k-regular graph on n vertices, with adjacency matrix A. Then the multiplicity of k as an eigenvalue of A is the number of connected components of G.*

Proof. The spectrum of A is the list of roots of the polynomial $\det(A - xI)$. This may be rewritten as

$$\det(A - kI_kI - xI) = \det((k - x)I - (kI - A)) = \det((k - x)I - L),$$

where L is the Laplacian matrix of A. Thus x is an eigenvalue of A if and only if $k - x$ is an eigenvalue of L with the same multiplicity. In particular the multiplicity of k as an eigenvalue of A is the multiplicity of 0 as an eigenvalue of L, which is the number of connected components of G, by Theorem **??**. □

Now let G be a connected strongly regular graph with parameters (n, k, λ, μ), and with adjacency matrix A. Then A^2_{uv} is the number of walks of length 2 from u to v in G, so

$$A^2_{uv} = \begin{cases} k & \text{if } \; u = v \\ \lambda & \text{if } \; uv \text{ is an edgeof } G \\ k & \text{if } \; u = v \\ \text{u and v are not adjacent in G} & \end{cases}$$

Thus

$$A^2 = kI + \lambda A + \mu(J - I - A) \implies A^2 - (\lambda - \mu)A - (k - \mu)I = \mu J.$$

Now let v be an eigenvector of A corresponding to an eigenvalue θ with $\theta \neq k$. Then v is orthogonal to $\mathbf{1}$ and so $Jv = 0$. Multiplying both sides of the above equation on the right by v gives

$$\theta^2 v - (\lambda - \mu)\theta v - (k - \mu)v = 0.$$

Since v is not the zero vector and since θ, λ and μ are all real numbers, this means that

$$\theta^2 - (\lambda - \mu)\theta - (k - \mu) = 0.$$

The roots of this quadratic equation are

$$\frac{(\lambda-\mu)\pm\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}.$$

For convenience we write Δ for the expression $(\lambda-\mu)^2+4(k-\mu)$. The eigenvalues of A then are

$$k, \theta_1 = \frac{(\lambda-\mu)+\sqrt{\Delta}}{2}, \theta_2 = \frac{(\lambda-\mu)-\sqrt{\Delta}}{2}.$$

Note that Δ is positive. It is clear that $k-\mu$ cannot be negative, since the number of neighbours of a vertex v of G cannot be fewer than its number of common neighbours with another vertex u. It is possible that $k-\mu=0$, this happens for example in the case of complete bipartite graphs. However if $k=\mu$ then λ cannot also be equal to μ, since an edge in a k-regular graph can belong to at most $k-$ triangles.

Since G is connected, k has multiplicity 1. Let m_1 and m_2 be the respective multiplicities of θ_1 and θ_2 as eigenvalues of A. Since the trace of A is 0, we have the following equations

$$m_1+m_2=n-1, \quad m_1\theta_1+m_2\theta_2+k=0.$$

Solving these equations gives

$$m_1=-\frac{(n-1)\theta_2+k}{\theta_1-\theta_2}, \; m_2=\frac{(n-1)\theta_1+k}{\theta_1-\theta_2}.$$

Note that $\theta_1-\theta_2=\sqrt{Delta}$. Entering the expressions for θ_1 and θ_2 in terms of the parameters of G to the equations above, we find

$$\begin{aligned} m_1 &= \frac{1}{2}\left[(n-1)-\frac{2k+(n-1)(\lambda-\mu)}{\sqrt{\Delta}}\right] \\ m_2 &= \frac{1}{2}\left[(n-1)+\frac{2k+(n-1)(\lambda-\mu)}{\sqrt{\Delta}}\right] \end{aligned}$$

Since $m_1+m_2=n-1$ it is clear that if one of m_1, m_2 is an integer then so is the other. That m_1 is an integer requires either that Δ is a square or that $2k+(n-1)(\lambda-\mu)=0$. In the latter case n must be odd.

Example 4.2.2. *Let G be the Petersen graph, with parameters $(10,3,0,1)$. Then*

$$k=3, \Delta=1+4(2)=9, \theta_1=1, \; \theta_2=-2, \; m_1=4, \; m_2=5.$$

We have shown that a connected strongly regular graph has exactly three distinct eigenvalues, k with multiplicity 1 and θ_1 and θ_2 with multiplicities adding to $n-1$. The product of θ_1 and θ_2 is non-positive (it can be zero in the case $k=\mu$), and θ_1 and θ_2 are distinct. We now show that a connected regular graph with exactly three distinct eigenvalues must be strongly regular.

Theorem 4.2.3. *Let G be a connected graph of order n that is regular of degree k, and let A be its adjacency matrix. Suppose that A has just three distinct eigenvalues: k, α and β. Then A is strongly regular.*

Proof. Since G is connected, A has k just once as an eigenvalue, with corresponding eigenvector $\mathbf{1}$. Every eigenvector of A corresponding to α or β is in the right nullspace of the matrix $A'=(A-\alpha I)(A-\beta I)$ (to see this note that $(A-\alpha I)$ and $(A-\beta I)$ commute with each other). Since the eigenspaces of A and B together account for the subspace of $\mathbb{R}^n$ of dimension $n-1$ consisting of all vectors orthogonal to $\mathbf{1}$. It follows that A' has rank 1 and that every row of A' has all entries equal. Finally

$$A'\mathbf{1}=(A-\alpha I)(A-\beta I)\mathbf{1}=(A-\alpha I)(k-\beta)\mathbf{1}=(k-\alpha)(k-\beta)\mathbf{1},$$

which means that

$$A' = (A - \alpha I)(A - \beta I) = \frac{1}{n(k-\alpha)(k-\beta)} J.$$

In particular then A^2 is a linear combination of A, I and J, hence of A, I and $J - I - A$. This means that A^2 has the same entry in every position on the diagonal, the same entry in all positions corresponding to edges of G, and the same entry in all positions corresponding to non-edges of G. Since A is a $(0,1)$-matrix, these entries are all non-negative integers and it follows that G is strongly regular. ☐

4.3 Two classes of strongly regular graphs

Let G is a strongly regular graph with parameters (n, k, λ, μ), and assume that $k \leqslant \frac{n-1}{2}$; there is no real loss of generality in this assumption since either G or its complement has this property. We have seen that the eigenvalues of G occur with multiplicities

$$1, m_1 = \frac{1}{2}\left[(n-1) - \frac{2k + (n-1)(\lambda - \mu)}{\sqrt{\Delta}}\right],\ m_2 = \frac{1}{2}\left[(n-1) + \frac{2k + (n-1)(\lambda - \mu)}{\sqrt{\Delta}}\right].$$

The condition that m_1 and m_2 are integers means that one of the following two cases occurs:

1. $2k + (n-1)(\lambda - \mu) \neq 0$ and Δ is an integer square; $m_1 \neq m_2$ in this case.
2. $2k + (n-1)(\lambda - \mu) \neq 0$, and $m_1 = m_2 = \frac{1}{2}(n-1)$ (this is referred to as the "half case" for this reason). In this case n must be odd obviously. Furthermore, since $2k \leqslant n - 1$, the condition that
$$2k = (n-1)(\mu - \lambda)$$
can be satisfied only if $2k = n - 1$ and $\mu - \lambda = 1$, so $\lambda = \mu - 1$ Moreover we know from Theorem **??** that $k(k - \lambda - 1) = (n - 1 - k)\mu$. Since $n - 1 - k = k$ and $\lambda + 1 = \mu$, this means that $k - \mu = \mu$ or $k = 2\mu$. Finally $n = 2k + 1 = 4\mu + 1$ and G has parameters
$$(4\mu + 1, 2\mu, \mu - 1, \mu)$$
for some positive integer μ. A strongly regular graph of this type is called a *conference graph.*

We look briefly at some examples of both types. The Kneser graph $Kn(n, 2)$ (the complement of the line graph of K_n) is an example of the first type. In the case $n = 5$, this is the Petersen graph which has parameters $(10, 3, 0, 1)$, with

$$\Delta = 1^2 + 4(3-1) = 9, \theta_1 = \frac{-1+3}{2} = 1, \theta_2 = \frac{-1-3}{2} = -2.$$

$$m_1 = \frac{1}{2}\left[9 - \frac{6 + 9(-1)}{3}\right] = 5,\ m_2 = \frac{1}{2}\left[9 + \frac{6 + 9(-1)}{3}\right] = 4.$$

In general the Kneser graph $Kn(n, 2)$ has parameters

$$\left(\binom{n}{2}, \binom{n-2}{2}, \binom{n-4}{2}, \binom{n-3}{2}\right).$$

Recall that for any integer m, $\binom{m+1}{2} - \binom{m}{2} = m$ (easily verified by a calculation or by a counting exercise). Thus $\lambda - \mu = 4 - n$ for the $Kn(n, 2)$, and $k - \mu = n - 3$. Then

$$\Delta = (4-n)^2 + 4(n-3) = n^2 - 8n + 16 + 4n - 12 = n^2 - 4n + 4 = (n-2)^2,$$

so Δ is a square. The eigenvalues are

$$\theta_1 = \frac{(4-n) + (n-2)}{2} = 1,\ \ \theta_2 = \frac{(4-n) - (n-2)}{2} = 3 - n.$$