Chapter 4

Strongly Regular Graphs

4.1 Parameters and Properties

Recall that a (simple, undirected) graph is *regular* if all of its vertices have the same degree. This is a strong property for a graph to have, and it can be recognized easily from the adjacency matrix, since it means that all row sums are equal, and that **1** is an eigenvector.

If a graph G of order n is regular of degree k, it means that kn must be even, since this is twice the number of edges in G. If $k \leqslant n-1$ and kn is even, then there does exist a graph of order n that is regular of degree k (showing that this is true is an exercise worth thinking about).

Regularity is a strong property for a graph to have, and it implies a kind of symmetry, but there are examples of regular graphs that are not particularly "symmetric", such as the disjoint union of two cycles of different lengths, or the connected example below.

Various properties of graphs that are stronger than regularity can be considered, one of the most interesting of which is strong regularity.

Definition 4.1.1. *A graph* G *of order* n *is called* strongly regular *with parameters* (n, k, λ, μ) *if*

- *every vertex of* G *has degree* k*;*
- *if* u *and* v *are adjacent vertices of* G*, then the number of common neighbours of* u *and* v *is* λ *(every edge belongs to* λ *triangles);*
- *if* u *and* v *are non-adjacent vertices of* G*, then the number of common neighbours of* u *and* v *is* μ*;*
- $1 \leqslant k < n-1$ *(so the complete graph and the null graph of* n *vertices are not considered to be strongly regular).*

So a srg (strongly regular graph) is a regular graph in which the number of common neighbours of a pair of vertices depends only on whether that pair forms an edge or not).

Examples

1. C_4 is strongly regular with parameters $(4, 2, 0, 2)$.
2. C_5 is strongly regular with parameters $(5, 2, 0, 1)$.
3. Apart from those two examples, C_n is not strongly regular: C_1, C_2 and C_3 are ruled out because they are complete graphs, and for $n \geqslant 6$, a pair of non-adjacent vertices may have either 1 common neighbour or none.

Strongly regular graphs are elusive and somewhat mysterious objects that have connections to various combinatorial constructions and to algebra over finite fields. A couple of slightly more complicated general families are described below.

Example 4.1.2. Recall that the *line graph* of K_n has vertices given by the $\binom{n}{2}$ edges of K_n, and a pair of vertices is adjacent if and only if the corresponding edges in K_n share a vertex.

- Let x be a vertex of $L(K_n)$, corresponding to the edge uv of K_n. The degree of x in $L(K_n)$ is the number of edges in K_n (other than uv) that are incident with either u or v. This is $2 \times (n-2) = 2n-4$.
- Suppose that x and y are adjacent vertices in $L(K_n)$, corresponding to the edges uv and uw of K_n. The number of common neighbours of x and y in $L(K_n)$ is the number of edges of K_n (other than uv and uw) that share a vertex with both uv and uw. There are $n-2$ of these: vw and the $n-3$ remaining edges involving u.
- Suppose that x and y are non-adjacent vertices in $L(K_n)$, corresponding to edges uv and wt of K_n. Then the number of common neigbours of x and y in $L(K_n)$ is the number of edges of K_n that are incident with one vertex in $\{u, v\}$ and one in $\{w, t\}$. There are four of these: uw, ut, vw and vt. So x and y have four common neighbours in $L(K_n)$.

The conclusion is that $L(K_n)$ is a strongly regular graph with parameters $\left(\binom{n}{2}, 2n-4, n-2, 4\right)$.

Example 4.1.3. Let $K_{n,n}$ denote the complete bipartite graph with n vertices in each part. The line graph $L(K_{n,n})$ has n^2 vertices, all of degree $2n-2$. If two of these vertices are adjacent, they have $n-2$ common neighbours. If two are non-adjacent, they have 2 common neighbours. So $L(K_{n,n})$ is a strongly regular graph with parameters $(n, 2n-2, n-2, 2)$.

Note that the complement of $L(K_n)$ is the *Kneser graph* $\mathrm{Kn}(n, 2)$. This is the graph whose vertices are the 2-element subsets of a set with n elements, and in which two vertices are adjacent if and only if the subsets that they represent are disjoint. The Kneser graph $\mathrm{Kn}(4, 2)$ consists of three isolated edges, and $\mathrm{Kn}(5, 2)$ is the famous *Petersen graph*. In general $\mathrm{Kn}(n, 2)$ is a strongly regular graph with parameters $\left(\binom{n}{2}, \binom{n-2}{2}, \binom{n-4}{2}, \binom{n-3}{2}\right)$.

It is true in general that the complement of a strongly regular graph is strongly regular and the relationship between their parameters can be figured out without too much trouble.

Theorem 4.1.4. *Let* G *be a strongly regular graph with parameters* (n, k, λ, μ). *Then* $\overline{G}$ *is a strongly regular graph with parameters* $(n, n-k-1, ,)$.

Proof. It is straightforward to observe that $\overline{G}$ has n vertices and is regular of degree $n-k-1$.

Let uv be an edge of $\overline{G}$. The number of triangles to which uv belongs in $\overline{G}$ is the number of vertices in G that are adjacent to neither u nor v. In G, uv is not an edge, each of u and v has k neighbours, and μ vertices are common neighbours of u and v. So the number of vertices that are adjacent to at least one of u and v is $2k-\mu$. Thus the number of vertices (other than u and v) that is adjacent to neither u nor v is $n-2-2k+\mu$. This is the number of triangles to which the edge uv belongs in $\overline{G}$.

Now suppose that u and v are non-adjacent edges in $\overline{G}$. Then uv is an edge of G. Each of u and v has $k-1$ additional neighbours in G, and they have λ common neighbours, so $2k-2-\lambda$ is the number of vertices (other than u and v themselves) that are adjacent in G to at least one of u and v. That leaves $n-2-(2k-2-\lambda)$ or $n-2k+\lambda$ vertices in G that are adjacent to neither u nor v. This is the number of common neighbours of u and v in $\overline{G}$.

We conclude that $\overline{G}$ is a strongly regular graph with parameters $(n, n-k-1, n-2-2k+\mu, n-2k+\lambda)$. ☐

Our final theorem in this section presents a compatibility condition on the parameters in a strongly regular graph. Not surprisingly, not every set of four non-negative integers is a candidate for being the set of parameters. After looking at the adjacency spectrum of a srg in the next section, we will obtain some more constraints of this nature.

Theorem 4.1.5. *Let* G *be a strongly regular graph with parameters* (n, k, λ, μ). *Then* $k(k-\lambda-1) = (n-k-1)\mu$.

Proof. We count, in two ways, the number of ordered triples of the form (u, v, w) in G with the property that v is adjacent to both u and w and that u and w are not adjacent to each other.

Suppose we choose v first - we have n choices here. Regardless of how this choice is made, the number of choices avaiable for a neighbour u of v is k. Having chosen u, the final step is to choose a vertex w that is adjacent to v but is not a common neighbour of u and v. There are $k-1$ neighbours of v from which w may be chosen, but λ of these are also neighbours of u. So the number of choices for w is $k-1-\lambda$. Hence the number of choices for the triple (u, v, w) is $nk(k-\lambda-1)$.

On the other hand suppose we choose u first. We have n choices for u, and then we may choose w from among the $n-k-1$ non-neighbours of u. Having done this we have μ choices for v among the common neighbours of u and w. So the number of choices for the triple (u, v, w) is $n(n-k-1)\mu$.

Putting these two counts together we find

$$nk(k-\lambda-1) = n(n-k-1)\mu \Longrightarrow k(k-\lambda-1) = (n-k-1)\mu.$$

□