3.3 Connectivity and the graph Laplacian

Let G be a connected graph of order n. If S is a set of vertices of G, then $G\backslash S$ is the graph obtained from G by deleting the vertices of G and their incident edges.

Definition 3.3.1. *A* vertex-cutset *of* G *is set* S *of vertices of* G *for which the graph* $G\backslash S$ *is disconnected. The* vertex connectivity *of* G*, denoted* $\kappa(G)$*, is defined to be the minimum number of vertices in a vertex-cutset.*

Notes

1. $\kappa(G) = 0$ if and only if G is disconnected.

2. If G is a tree on more than 2 vertices, then $\kappa(G) = 1$.

3. If $\kappa(G) = 1$, it means that G has vertices x and y with the property that every path from x to y goes via a particular vertex v (a *cut vertex*.

4. If $\kappa(G) = k$ and S is a vertex-cutset with k elements, it means that there is a pair of vertices x and y in G (not in S) with the property that every path from x to y in G is via a vertex of S (but there is no set of fewer than k vertices for which there exists such a pair).

5. G is said to be t-vertex-connected if it cannot be disconnected by the deletion of fewer than t vertices. So if $\kappa(G) = t$, it means that G is t-vertex-connected but not $(t+1)$-vertex-connected.

6. The complete graph K_n cannot be disconnected by the deletion of vertices, so its vertex connectivity is not defined.

7. If G is a non-complete graph of order n, then its vertex connectivity is at most $n-2$, since it can be disconnected by the removal of all vertices except for some non-adjacent pair.

Definition 3.3.2. *An* edge cutset *of a connected graph* G *is a set of edges of* G *whose deletion would disconnect* G*. The* edge-connectivity *of* G*, denoted* $\epsilon(G)$*, is the minimum number of edges in an edge-cutset.*

Notes

1. A single edge whose deletion would disconnect G is called a *bridge* or a *cut edge*. If $e = uv$ is a bridge in G, then e is the unique path between u and v in G.

2. If $\epsilon(G) = 2$ and $\{e_1, e_2\}$ is an edge-cutset in G, it means that every cycle in G that contains e_1 also contains e_2, or that every cycle containing e_2 also contains e_1.

3. In any connected graph G, let $\delta(G)$ be the minimum of the vertex degrees in G, and let v be a vertex with $\deg(v) = \delta$. Then G can be disconnected by the deletion of the δ edges incident with v, and so $\epsilon(G) \leqslant \delta$.

4. It follows that the edge connectivity of a non-complete graph of order n is at most $n-2$.

Lemma 3.3.3. *Let* G *be a (non-complete) connected graph on* n *vertices. Then* $\kappa(G) \leqslant \epsilon(G)$.

Proof. Let $m = \epsilon(G)$ (note $m \leqslant n-2$) and suppose that $S = \{e_1, \dots, e_m\}$ is a set of edges of G whose deletion disconnects G. Since the removal of a single edge can break a connected graph into at most two components, and since S is a minimal set whose deletion disconnects G, it follows that $G\backslash S$ has exactly two connected components. Furthermore, since the restoration of any of the edges e_i would reconnect $G\backslash S$, it follows that for each i, the vertices x_i and y_i of e_i belong to different components of $G\backslash S$. We may label these vertices so that $x_1, \dots, x_m$ all belong to the component C_1 of $G\backslash S$, and $y_1, \dots, y_m$ all belong to the other component C_2 (note that the x_i are not necessarily distinct, same for the y_i).

If the x_i are not all of the vertices of C_1, then deleting the vertices x_i and their incident edges disconnects G. Since the number of x_i is at most m, this shows that $\kappa(G) \leqslant m$.

On the other hand, if $\{x_1, \ldots, x_m\}$ is the full vertex set of C_1, then deletion of this set may not disconnect G. In this case the only possible neighbours of x_1 in G are those x_j for which $x_j \neq x_1$ and those y_l for which $x_l = x_1$. The total number of such vertices is at most m, and their deletion (along with their incident edges) leaves a graph G' in which the vertex x_1 is isolated. Since $m \leqslant n - 2$, the graph G' possesses at least two more vertices, hence it is disconnected and $\kappa(G) \leqslant m$. □

Now we return to the Laplacian spectrum of a graph. Let G be a graph of order n with Laplacian matrix $L(G)$, and let $0, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $L(G)$, in increasing order. We know that $\lambda_2 > 0$ if and only if G is connected. We show now that λ_2 is bounded above by $\kappa(G)$. To see this we need the following lemma. In the statement of this lemma, the column vector x is considered as a real-valued function on the vertex set of G, its value on the vertex u is the component X_u.

Lemma 3.3.4. *Let* G *be a graph with Laplacian matrix* L. *Let* $x \in \mathbb{R}^n$. *Then*

$$x^T L x = \sum_{uv \in E(G)} (x_u - x_v)^2.$$

Proof. Let B be an oriented incidence matrix for G, and recall that $L = B^T B$. Then

$$x^T L x = x^T B^T B x = (Bx)^T (Bx) = (Bx) \cdot (Bx).$$

The rows of B (and hence the entries of Bx) are labelled by the edges of G. If $e = uv$ is an edge of G, then the entry of Bx in the position corresponding to e is $\pm(x_u - X_v)$. Thus

$$x^T L x = \sum_{uv \in E(G)} (x_u - x_v)^2.$$

□

We now show that $\lambda_2(G)$ is the minimum over all unit vectors x satisfying $x \perp \mathbf{1}$ of the expression $\sum_{uv \in E(G)} (x_u - x_v)^2$.

Theorem 3.3.5. *Let* G *be a graph of order* n *with Laplacian spectrum* $0 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. *Then* λ_2 *is the minimum over all unit vectors* x *that belong to* $\mathbf{1}^\perp$ *of the expression* $x^T L(G) x$, *and this minimum is attained if and only if* x *is an eigenvector of* $L(G)$ *corresponding to* λ_2.

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $L(G)$, where $v_1 = \frac{1}{\sqrt{n}}\mathbf{1}$ and otherwise v_i corresponds to the eigenvalue λ_i. Let x be a unit vector in $\mathbb{R}^n$ that is orthogonal to $\mathbf{1}$. Then

$$x = a_2 v_2 + \cdots + a_n v_n$$

for some real numbers a_i with $\sum a_i^2 = 1$.

$$\begin{aligned} x^T L(G) x &= (a_2 v_2 + \cdots + a_n v_n)^T (a_2 \lambda_2 v_2 + \cdots + a_n \lambda_n v_n) \\ &= \lambda_2 a_2^2 + \cdots + \lambda_n a_n^2 \\ &= \lambda_2 (a_2^2 + \cdots + a_n^2) + \sum_{i=3}^{n} (\lambda_i - \lambda_2) a_i^2 \\ &= \lambda_2 + \sum_{i=3}^{n} (\lambda_i - \lambda_2) a_i^2. \end{aligned}$$

Since $\lambda_i - \lambda_2$ is non-negative whenever $i \geqslant 3$, the last line above says that $x^T L(G) x \geqslant \lambda_2$ and that equality occurs here if and only if $a_i = 0$ whenever $\lambda_i > \lambda_2$, which means that x is an eigenvector of $L(G)$ corresponding to λ_2. □

Lemma 3.3.4 and Theorem 3.3.5 together give us the necessary ingredients to prove that $\lambda_2(G)$ is bounded above by the vertex connectivity of G.

Theorem 3.3.6. *Let* G *be a (non-complete) connected graph. Then* $\lambda_2(G) \leqslant \kappa(G)$.

Proof. Write k for $\kappa(G)$ and let S be a set of vertices of G whose deletion disconnects G. Since the graph $G\backslash S$ is disconnected, $\lambda_2(G\backslash S) = 0$, and there exists a unit vector x' in $\mathbb{R}^{n-k}$ that is a zero eigenvector of $L(G\backslash S)$ and is orthogonal to $\mathbb{1}_{n-k}$. Let x be the unit vector in $\mathbb{R}^n$ that coincides with x' on the vertices of $G\backslash S$ and has zeros in the positions corresponding to vertices of S. By Theorem 3.3.5

$$\lambda_2(G) \leqslant x^T L(G) x = \sum_{uv \in E(G)} (x_u - x_v)^2.$$

We now break this sum into separate components, one involving the edges of $G\backslash S$ and one involving edges for which one vertex is in S and the other is not. There is no need to consider edges whose vertices are both in S since $x_w = 0$ whenever $w \in S$. Thus

$$\sum_{uv \in E(G)} (x_u - x_v)^2 \quad \leqslant \quad \sum_{uv \in E(G\backslash S)} (x_u - x_v) + \sum_{u \in S} \sum_{v \notin S} x_v^2.$$

Since x' is a zero eigenvector of $L(G\backslash S)$ (which means x is constant on the components of $G\backslash S$), the term $\sum_{uv \in E(G\backslash S)} (x_u - x_v)$ is zero. The other term is bounded above by

$$\sum_{u \in S} \sum_{v \in V(G)} x_v^2 = \sum_{u \in S} ||x|| = |S| = k.$$

Thus $\lambda_2(G) \leqslant k$ as required. □

Definition 3.3.7. *For a connected graph* G, $\lambda_2(G)$ *is called the* algebraic connectivity *of* G.

The following example shows that $\lambda_2(G)$ may be equal to the vertex connectivity of G, or quite far away from it.
Example For the cycle C_n of length n, $\lambda_2(C_n) = 2 - 2\cos(\frac{2\pi}{n})$. So $\lambda_2(C_4) = 2$ which is equal to the vertex connectivity of C_4. For all $n \geqslant 4$, $\kappa(C_n) = 2$. However, as $n \to \infty$, $\lambda_2(C_n) \to 0$. This reflects the fact that as n increases, C_n becomes more "flimsily" vertex-connected - to disconnect it, an ever decreasing proportion of its vertices needs to be removed. This observation is a heuristic not a theorem - there are several classes of graphs for which the algebraic connectivity behaves in this manner. Graphs with low values of λ_2 tend to have high diameter compared to their order.

Despite this remark another conclusion from looking at the graphs C_n is that the "gap" between the algebraic connectivity and the vertex connectivity of a graph is not always easy to interpret. Another way of looking at the meaning of λ_2 is to consider how it changes when adjustments are made to a graph. The step of adding an edge between two non-adjacent vertices in a graph cannot decrease the vertex connectivity, and can leave it unchanged or increase it by 1. Our next theorem shows that something like this is true for the algebraic connectivity.

Theorem 3.3.8. *Let* G *be a (non-complete) graph of order* n, *and let* H *be obtained from* G *by adding an edge between two non-adjacent vertices. Then*

$$\lambda_2(G) \leqslant \lambda_2(H) \leqslant \lambda_2(G) + 2.$$

Proof. Let r and s be the two vertices that are adjacent in H and not in G. Let x be a unit vector in $\mathbb{R}^n$, orthogonal to **1**, for which

$$\lambda_2(H) = \sum_{uv \in E(H)} (x_u - x_v)^2 = \sum_{uv \in E(G)} (x_u - x_v)^2 + (x_r - x_s)^2.$$

Since $\lambda_2(G)$ is the minimum over all unit vectors y (orthogonal to **1** of the sum of $(y_u - y_v)^2$ over all edges uv of G, it follows that $\lambda_2(G)$ is at most equal to $\sum_{uv \in E(H)} (x_u - x_v)^2$ and in particular $\lambda_2(G) \leqslant \lambda_2(H)$.

On the other hand, let z be a unit eigenvector of $L(G)$ corresponding to $\lambda_2(G)$. Then

$$\lambda_2(H) \leqslant \sum_{uv \in E(G)} (z_u - z_v)^2 + (z_x - z_y)^2.$$

Note that

$$(z_x + z_y)^2 = z_x^2 + z_y^2 + 2z_x z_y \geqslant 0 \implies -2z_x z_y \leqslant z_x^2 + z_y^2.$$

Furthermore, since z_+x and z_y are components of a unit vector, we know that $z_x^2 + z_y^2 \leqslant 1$. Thus

$$0 \leqslant (z_x - z_y)^2 = -2z_x z_y + (z_x^2 + z_y^2) \leqslant 1 + 1 = 2.$$

Thus

$$\lambda_2(H) \leqslant \sum_{uv \in E(G)} (z_u - z_v)^2 + (z_x - z_y)^2 \implies \lambda_2(H) \leqslant \lambda_2(G) + 2.$$

□