3.2 Spanning Trees

A *tree* is a connected graph with no cycle. Some basic properties of trees are noted below.

- A tree of order n has exactly $n-1$ edges.
- Every tree has at least two *leaves*, i.e. vertices of degree 1.
- A graph is a tree if and only if it contains a *unique* path between any pair of its vertices.
- Trees can be regarded as minimally connected in the sense that the deletion of any edge would result in a disconnected graph.
- If the maximum degree of a vertex in a tree T is Δ, then T has at least Δ leaves.

Definition 3.2.1. *Let G be a connected graph. A* spanning tree *of G is a subgraph T which is a tree and whose vertex set is the full vertex set of G.*

Every connected graph has at least one spanning tree, since one may be obtained by repeating the step of deleting an edge that belongs to a cycle until none remain. Such a step will never disconnect a graph.

We also introduce the following piece of matrix notation: if u is a vertex of a graph G with Laplacian matrix $L(G)$, we denote by $L(G)[u]$ the matrix obtained from $L(G)$ by deleting the row and column corresponding to u. If G has order n, then $L(G)[u]$ is a principal $(n-1)\times(n-1)$ submatrix of $L(G)$. Similarly, if u and v are both vertices of G, we denote by $L(G)[u,v]$ the principal $(n-2)\times(n-2)$ submatrix of $L(G)$ obtained by deleting the rows and columns labelled by u and v. The main theorem of this section is the statement that, for any vertex u, $\det(L(G)[u])$ counts the spanning trees in G.

We have noted this already in the case of complete graphs and stars, and we also remark that if G is disconnected then L has rank at most $n-2$, so $L[u]$ is singular for all u, which is consistent with the theorem since the number of spanning trees in a disconnected graph is zero.

Theorem 3.2.2. *Let G be a graph with Laplacian matrix L. Let u be any vertex of G. Then $\det(L[u])$ is the number of spanning trees in G.*

The mechanism of the proof is induction on the number of edges. The induction step relies of the following two distinct methods of moving from G to a graph with one fewer edge.

- Let e be an edge of G. Then $G\backslash e$ is the graph obtained from G by deleting e (but not the vertices with which e is incident). Note that $G\backslash e$ need not be connected even if G is.
- Let e be an edge of G. Then G/e is obtained from G by *contracting* the edge e, which means identifying the two vertices of e together. If $e=uv$ then u and v are identified as a single vertex (still called either u or v) of G/e, and all neighbours of u or v in G are neighbours of this merged vertex in G/e. Note that if u and v have common neighbours in G, then G/e has multiple edges. If G is connected, then G/e is also connected.
- The number of edges in both $G\backslash e$ and G/e is one less than the number in G.

For any graph G, we denote the number of spanning trees of G by $\tau(G)$. The proof of Theorem 3.2.2 is presented as a series of lemmas.

Lemma 3.2.3. *Let G be a (connected) graph and let e be an edge of G. Then*

$$\tau(G)=\tau(G\backslash e)+\tau(G/e).$$

Proof. Every spanning tree of G that contains e contracts to a spanning tree of G/e when the edge e is contracted, and every spanning tree of G/e may be expanded to a spanning tree of G by reintroducing the edge e. Thus the number of spanning trees of G that contain e is $\tau(G/e)$.

On the other hand any spanning tree of G that does not contain e is a spanning tree of $G\backslash e$.

Since every spanning tree of G either contains e or does not contain e, it follows that

$$\tau(G) = \tau(G/e) + \tau(G\backslash e),$$

as required. □

The next lemma deals with the relationship between the Laplacian matrices of G, G/e and $G\backslash e$. For a graph G and a designated ordering of its vertices, we denote by E_{vv} the matrix that has a 1 in the row and column labelled by the vertex v, and zeros elsewhere.

Lemma 3.2.4. *Let* G *be a graph and let Now write* $e = uv$. *Then*

$$L(G)[u] = L(G\backslash e)[u] + E_{vv}.$$

Proof. The Laplacian matrix of $G\backslash e$ differs from that of G only in the entries in positions (u, u), (v, v), (u, v) and (v, u). Three of these entries, since they belong to the row and column labelled by u, are absent from $L(G)[u]$ and from $L(G\backslash e)[u]$. The fourth represents the degree of v, which is greater by one in G than in $G\backslash e$. Hence the $(n-1) \times (n-1)$ matrices $L(G)[u]$ and $L(G\backslash e[u])$ are related by the equation

$$L(G)[u] = L(G\backslash e)[u] + E_{vv}.$$

□

The next lemma relates $(n-2) \times (n-2)$ principal submatrices of $L(G)$ to those of $L(G/e)$. In the statement of the lemma, we interpret that when the edge $e = uv$ is contracted to form G/e, it is the vertex u that is "absorbed" into v and the vertex v that survives.

Lemma 3.2.5. *Let* G *be a graph and let* $e = uv$ *be an edge of* G. *Then*

$$L(G)[u, v] = L(G/e)[v].$$

Proof. Let w and z be vertices of G (other than u and v). If $w \neq z$, the entry in the (w, z) position of $L(G)[u, v]$ is -1 or 0 according as w is adjacent to z or not. Since w and z are adjacent in G if and only if they are adjacent in G/e, the (w, z)-entry of $L(G/e)[v]$ is the same as that of $L(G)[u, v]$. The (w, w)-entry of $L(G)[u, v]$ is the degree in G of w. This is the same as the degree of w in G/e (bearing in mind that a double edge from w to v in G/e contributes 2 to the degree of w in this graph). So the diagonal entries of $L(G)[u, v]$ also coincide with those of $L(G/e)[v]$. □

We are now in a position to complete the proof of Theorem 3.2.2, by induction on the number of edges.

Base: If G is a graph with a single edge, then the number of spanning trees of G is 1 if G has order 2 and zero if the order of G exceeds 2. If the order of G is 2, then $L(G) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and every 1×1 principal submatrix of $L(G)$ has determinant 1. If the order of G is 3 or greater, then $L(G)$ has one 2×2 prinicipal submatrix equal to $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and is otherwise full of zeros. In this case $L(G)$ has rank 1 and all of its $(n-1) \times (n-1)$ principal submatrices have determinant zero, which is the number of spanning trees in G.
So the theorem is true for graphs with one edge.

Induction hypothesis: We consider a particular graph G and assume that the theorem holds for all graphs with fewer edges than G (in particular for $G\backslash e$ and G/e for any edge e of G).

Induction step: Choose a vertex u of G. We need to show that $\det L(G)[u] = \tau(G)$. If u is isolated in G, then $\tau(G)$ and $\det L(G)[u]$ are both equal to zero and the theorem holds. If not, then there is an edge $e = uv$ in G. By Lemma 3.2.3,

$$\tau(G) = \tau(G\backslash e) + \tau(G/e).$$

By the induction hypothesis,

- $\tau(G\backslash e) = \det(L(G\backslash e)[u])$, and
- $\tau(G/e) = \det(L(G/e)[v])$.

From Lemma 3.2.5 we have $L(G/e)[v] = L(G)[u, v]$. From Lemma 3.2.4 we have

$$L(G)[u] = L(G\backslash e)[u] + E_{vv}.$$

Consider applying a determinant calculation using cofactor expansion by the Row of v to the above description of $L(G)[u]$. Where C_{vw} denotes the cofactor of the entry in the (v, w)-position of $L(G)[u]$, we have

$$\begin{aligned}
\det(L(G)[u]) &= \sum_{w \in V(G)\backslash\{u\}} (L(G)[u])_{vw} C_{vw} \\
&= \sum_{w \neq u,v} L(G\backslash e)[u]_{vw} C_{vw} + (L(G)[u])_{vv} C_{vv} \\
&= \sum_{w \neq u} L(G\backslash e)[u]_{vw} C_{vw} + C_{vv},
\end{aligned}$$

where the last line is an application of Lemma 3.2.4. Finally we have

$$\begin{aligned}
\sum_{w \neq u} L(G\backslash e)[u]_{vw} C_{vw} + C_{vv} &= \det(L(G\backslash e)[u]) + \det(L(G)[u, v]) \\
&= \tau(G\backslash e) + \tau(G/e),
\end{aligned}$$

by the induction hypothesis. Application of Lemma 3.2.3 now completes the proof.

Corollary 3.2.6. *The number of spanning trees of the complete graph* K_n *is* n^{n-2}.

Proof. Since $\text{spec}(L(K_n)) = [0, \underbrace{n, \dots, n}_{n-1}]$, the sum of all of the $(n-1) \times (n-1)$ principal minors of $L(K_n)$ is n^{n-1}. Since it follows from Theorem 3.2.2 that all of these n principal minors are equal, each of them is equal to n^{n-2}. □

In fact, a slighty stronger statement than Theorem 3.2.2 can be proved. Not only are all of the principal $(n-1) \times (n-1)$ principal minors of $L(G)$ equal to $\tau(G)$, but the cofactor of *every entry* of $L(G)$ is equal to $\tau(G)$.

For a square matrix A the *cofactor* C_{ij} of the entry A_{ij} is the $(-1)^{i+j} \times \det(A[Ri, Cj])$, where $A[Ri, Cj]$ is the matrix obtained by deleting Row i and Column j form A. The *adjugate* of A, denoted adj$A$, is the transpose of the matrix of cofactors of A, i.e. it is the matrix whose (i, j)-entry is A_{ji}. The relationship between A and its adjugate is

$$A\text{adj}(A) = \det(A) I_n = \text{adj}(A) A.$$

If A is invertible, this means that $A^{-1} = \frac{1}{\det A}\text{adj}(A)$.

Lemma 3.2.7. *Let* G *be a graph with Laplacian matricx* $L(G)$*. Then for all* (i, j)*, the cofactor* C_{ij} *of* $L(G)$ *is equal to* $\tau(G)$*.*

Proof. First suppose that G is disconnected. Then $\tau(G) = 0$, and the rank of $L(G)$ is less than $n-1$, so all $(n-1) \times (n-1)$ minors of $L(G)$ are zero. Thus all cofactors of $L(G)$ are equal to zero which is $\tau(G)$.

If G is connected, then since $L(G)$ is singular we know that $L(G)\text{adj}(L(G)) = 0_{n \times n}$. Furthermore, $L(G)$ has rank $n-1$ which means that the right nullspace of $L(G)$ is 1-dimensional, spanned by the all-1 vector $\mathbf{1}$. Thus every column of $\text{adj}(L(G))$ is a scalar multiple of $\mathbf{1}$. Since $L(G)$ is symmetric, so also is $\text{adj}(L(G))$, so every row of $\text{adj}(L(G))$ is a scalar multiple of $\mathbf{1}^T$. Since the diagonal entries of $\text{adj}(L(G))$ are all equal to $\tau(G)$, it follows that every entry of $\text{adj}(L(G))$ is equal to $\tau(G)$, as required. □