2.3 Connections to the adjacency spectrum

In this section we consider some of the consequences for spectral graph theory of the properties of symmetric and positive semidefinite real matrices that were established in Section 2.2. First we consider the connection between the adjacency spectrum of a regular graph and that of its complement.

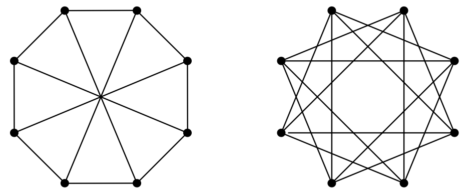
Recall that a graph is *regular* if all of its vertices have the same degree. Also, the complement of a graph G is the graph $\overline{G}$ that has the same vertex set as G and whose edges are precisely the non-edges of G. The adjacency matrix of $\overline{G}$ has 1s precisely in the off-diagonal positions where the adjacency matrix of G has zeros. So the adjacency matrices of any graph and its complement are related by the equation

$$A(G) + A(\overline{G}) + I_n = J,$$

where as usual J denotes the matrix in which every entry is 1.

If G is regular of degree k, then $\overline{G}$ is regular of degree $n - 1 - k$, where n is the order (number of vertices) of G.

Example



Note that if G is a k-regular graph, then k is an eigenvalue of $A(G)$, corresponding the eigenvector whose entries are all 1. This follows from the fact that the sum of the entries in each row of $A(G)$ is k.

Theorem 2.3.1. *Let G be a k-regular graph of order n, and let the spectrum of $A(G)$ be $[k, \theta_2, \dots, \theta_n]$. Then the spectrum of $A(\overline{G})$ is $[n - k - 1, -1 - \theta_2, \dots, -1 - \theta_n]$. Furthermore $A(G)$ and $A(\overline{G})$ have the same eigenvectors.*

Outline of proof: By Theorem 2.2.2 we may choose a basis $\{v_1, \dots, v_n\}$ consisting of $\mathbb{R}^n$ of mutually orthogonal eigenvectors of $A(G)$, where v_1 is the all-1 vector (corresponding to the eigenvalue k) and v_i corresponds to θ_i for $i \geqslant 2$. Note that $v_i \perp v_1$, in particular this means that $Jv_i = 0$ for $i \geqslant 2$. Now consider the product

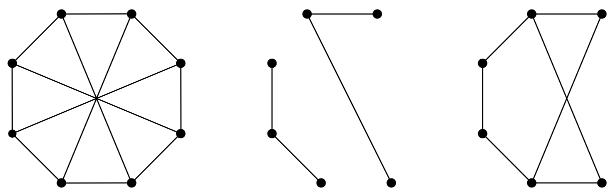
$$A(\overline{G})v_i = (J - A(G) - I_n)v_i.$$

Note: this is Problem 9 in Problem Sheet 1.

Now we consider how the adjacency spectrum of a graph G relates to the spectra of some of some of its subgraphs. We recall some definitions.

Definition 2.3.2. *Let G be a graph with vertex set V and edge set E. A* subgraph *of G is a graph whose vertex set is a subset of V and whose edge set is a subset of E. An* induced subgraph *of G is a subgraph H whose edge set consists of all edges of G that involve two vertices of H.*

Example A graph G, a (non-induced) subgraph H_1 and an induced subgraph H_2.



G H_1 H_2

If H is an *induced subgraph* of a graph G, then the adjacency matrix of H consists of the entries of the rows and columns of A(G) that label those vertices that belong to H. These form a *principal submatrix* of A(G). In general a principal submatrix of a sqaure matrix M is a square submatrix whose main diagonal coincides with that of M. In the case of adjacency matrices, principal submatrices correspond to induced subgraphs. If H′ is *any* subgraph of G, then the adjacency matrix of H is obtained from the relevant principal submatrix of A(G) by possible replacing some symmetrically opposite pairs of 1s with zeros. The adjacency matrices of the graphs G, H_1 and H_2 (corresponding to a vertex labelling that starts in the top right and proceeds anticlockwise) are given below.

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix},\ A(H_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},\ A(H_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The following lemma notes a useful property of positive semidefinite matrices.

Lemma 2.3.3. *Suppose that* A *is a symmetric PSD matrix. Then every principal submatrix of* A *is PSD. (If* A *is positive definite than every principal submatrix of* A *is positive definite).*

Proof. Let A_1 be the principal submatrix of A consisting determined by Rows and Columns $i_1, i_2, \dots, ik$. Let V be the subspace of $\mathbb{R}^n$ consisting of all those column vectors that have zeros outside of positions $i_1, \dots, i_k$. Then for every $v \in V_i$, let v_1 denote the vector in $\mathbb{R}^k$ whose entries are the entries from positions $i_1, \dots, i_k$ of V. Then for $v \in V$ we have

$$v^T A v = v_1^T A_1 v_1.$$

Since $v^T A v \geqslant 0$ for all $v \in V$, it follows that $v_1^T A_1 v_1 \geqslant 0$ for all $v_1 \in R^k$, hence that A_1 is positive semidefinite as required. □

Note that a particular consequence of Lemma 2.3.3 is that the diagonal entries of a symmetric PSD matrix must be non-negative (positive if the matrix is PD).

For any simple undirected graph G, the eigenvalues of A(G) are real numbers. We write $\lambda_{max}(G)$ for the maximum eigenvalue of G, and $\lambda_{min}(G)$ for the minimum eigenvalue of G. The following theorem is related to Lemma 2.3.3.

Theorem 2.3.4. *Let* H *be an induced subgraph of order* k *of a graph* G *of order* n*. Then*

$$\lambda_{min}(G) \leqslant \lambda_{min}(H) \leqslant \lambda_{max}(H) \leqslant \lambda_{max}(G).$$

Proof. Write A for the adjacency matrix of G and θ and μ respectively for the maximum and minimum eigenvalues of A. Suppose that σ is an eigenvalue of the matrix $\theta I_n - A$. Then

$$(\theta I - A)v = \sigma v \Longrightarrow Av = (\theta - \sigma)v$$

for some non-zero $v \in \mathbb{R}^n$. So the eigenvalues of $\theta I_n - A$ are obtained by subtracting the eigenvalues of A from θ, hence they are all non-negative and $\theta I_n - A$ is positive semidefinite. It follows that $\theta I_k - A(H)$ is positive semidefinite, since it is a principal submatrix of $\theta I_n - A$. This means that $\theta - \rho \geqslant 0$ for every eigenvalue ρ of A(H), so $\lambda_{max}(H) \leqslant \theta$.

On the other hand every eigenvalue of $A - \mu I$ is of the form $\sigma - \mu$ for some eigenvalue σ of A, and is therefore non-negative. Thus $A - \mu I_n$ is a positive semidefinite matrix and so is its principal submatrix $A(H) - \mu I_k$, which means that $\rho - \mu \geqslant 0$ for every eigenvalue ρ of A(H), and in particular $\lambda_{min}(A(H)) \geqslant \mu$. □

In fact it is true for any subgraph H of G that

$$\lambda_{\min}(G) \leqslant \lambda_{\min}(H) \leqslant \lambda_{\max}(H) \leqslant \lambda_{\max}(G),$$

but to prove this for non-induced subgraphs requires the Perron-Frobenius Theorem. More on this later.