Thus
$$\hat{T}(\hat{b_j}) = \sum_i a_{ji}\hat{b_i},$$

and the matrix of $\hat{T}$ with respect to $\hat{\mathcal{B}}$ is $A^T$, the transpose of the matrix of $T$ with respect to $\mathcal{B}$.

**Remark:** *Natural isomorphism between* $V$ *and its double dual* $\hat{\hat{V}}$.
We have seen above that every finite dimensional vector space $V$ has the same dimension as its dual $\hat{V}$, and hence they are isomorphic as vector spaces. Once we choose a basis for $V$ we also define a dual basis for $\hat{V}$ and the correspondence between the two bases gives us an explicit isomorphism between $V$ and $\hat{V}$. However this isomorphism is not intrinsic to the space $V$, in the sense that it depends upon a choice of basis and cannot be described independently of a choice of basis.

The dual space of $\hat{V}$ is denoted $\hat{\hat{V}}$; it is the space of all linear mappings from $\hat{V}$ to $\mathbb{F}$. By all of the above discussion it has the same dimension as $\hat{V}$ and $V$. The reason for mentioning this however is that there *is* a "natural" isomorphism $\theta$ from $V$ to $\hat{\hat{V}}$. It is defined in the following way, for $x \in V$ and $f \in \hat{V}$ - note that $\theta(x)$ belongs to $\hat{\hat{V}}$, so $\theta(x)(f)$ should be an element of $\mathbb{F}$.

$$\theta(x)(f) = f(x).$$

To see that $\theta$ is an isomorphism, suppose that $x_1, \dots, x_k$ are independent elements of $V$. Then $\theta(x_1), \dots, \theta(x_k)$ are independent elements of $\hat{\hat{V}}$. To see this let $a_1, \dots, a_k$ be element of $\mathbb{F}$ for which $a_1\theta(x_1) + \cdots + a_k\theta(x_k) = 0$. This means that $f(a_1x_1 + \cdots + a_kx_k) = 0$ for all $f \in \hat{V}$, which means that $a_1x_1 + \cdots + a_kx_k = 0$, which means that each $a_i = 0$.

## 1.3 Matrices and Graphs

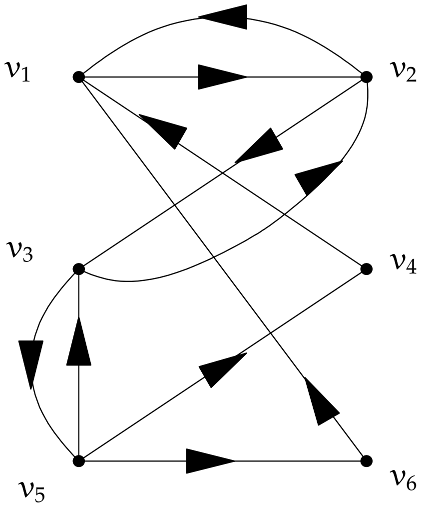
A *directed graph* (or *digraph*) $G$ consists of a non-empty set $V$ of vertices and a set $E$ of ordered pairs of these vertices, called edges. Each edge is directed from one vertex of $G$ to another. An *undirected graph* is similar, except that the edges are not considered to have a direction.

A number of square matrices are typcially associated to a graph, the most elementary of which is the *adjacency matrix*.

**Definition 1.3.1.** *Let* $G$ *be a (directed) graph with* $n$ *vertices labelled* $v_1, \dots, v_n$. *The* adjacency matrix $A$ *of* $G$ *is the* $n \times n$ *matrix whose entries are given by*

$$A_{ij} = \begin{cases} 1 & \text{there is an edge directed from } v_i \text{ to } v_j \text{ in } G \\ 0 & \text{otherwise} \end{cases}$$

**Example 1.3.2.** *A directed graph and its adjacency matrix.*




$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If $u$ and $v$ are vertices in a directed graph, a *(directed) walk* from $u$ to $v$ is a sequence of vertices that starts at $u$ and finishes at $v$, with the property that every pair of consecutive entries is a directed edge. The length of a path is the number of edges in it. In the example above, $v_5, v_6, v_1, v_2, v_1, v_2, v_3$ is a directed walk of length 6 from $v_5$ to $v_3$. The adjacency matrix has the interesting property that its powers count directed walks.

**Theorem 1.3.3.** *Let $G$ be a directed graph with adjacency matrix $A$ (with respect to the ordering $v_1, v_2, \dots, v_n$ of the vertices). For every positive integer $k$, the $(i, j)$ entry of $A^k$ is the number of walks of length $k$ from $v_i$ to $v_j$ in $G$.*

*Proof.* By induction on $k$. The case $k = 1$ is just the definition of the adjacency matrix.
So assume that the theorem is true for $k = m - 1$ and consider $k = m$. Then

$$(A^m)_{ij} = \sum_l A_{il} A_{lj}^{m-1}.$$

Every path of length $m$ from $v_i$ to $v_j$ must start with an edge from $v_i$ to some $v_l$ and follow that with a path of length $m - 1$ from $v_l$ to $v_j$. For each $l$, the entry $A_{il}$ is 1 if $(v_i, v_l)$ is an edge and 0 otherwise. By the induction hypothesis $A_{lj}^{m-1}$ is the number of directed walks of length $m - 1$ from $v_l$ to $v_j$ in $G$. Thus for each vertex $l$, the integer $A_{il} A_{lj}^{m-1}$ is the number of walks of length $m$ from $v_i$ to $v_j$ that have $v_l$ as their second vertex. The sum over $l$ of these is the total number of walks of length $m$ from $v_i$ to $v_j$ in $G$. This completes the induction proof. □

An *undirected* graph resembles a directed graph except that the edges are *unordered* pairs of vertices. The adjacency matrix of an undirected graph is symmetric.

Note that the adjacency matrix of a (directed or undirected) graph $G$ depends not only on the graph itself but also on the choice of an ordering of the vertices. Suppose that $\sigma$ is a permutation of the set $\{1, \dots, n\}$. Let $A$ be the adjacency matrix of $G$ with respect to the ordering $v_1, \mathtt{dots}, v_n$ of the vertices, and let $A'$ be the adjacency matrix with respect to the ordering $v_{\sigma(1)}, \dots, v_{\sigma(n)}$. Then $A'$ is obtained from $A$ by

- first reordering the columns by replacing Column 1 with Column $\sigma(1)$, Column 2 with Column $\sigma(2)$, and so on. This means multiplying on the right by the matrix $P_\sigma$, in which each Column $j$ (for each $j$) has a 1 as its $\sigma(j)$-th entry and is otherwise full of zeros.

- then reordering the rows by replacing Row 1 with Row $\sigma(1)$, Row 2 with Row $\sigma(2)$, etc. This means multiplying $A$ on the left by the matrix $(P_\sigma)^T$, which is also equal to $P_\sigma^{-1}$.

- A *permutation matrix* is a matrix that has exactly one 1 in each row and column and is otherwise full of zeros. Every permutation matrix has the property that its inverse is equal to its transpose. We have shown that adjacency matrices $A$ and $A'$ represent the same graph if and only if
$$A' = P^T A P,$$
for some permutation matrix $P$. This relation is known as *permutation equivalence*; it is a special case of both similarity and congruence.

The adjacency matrix is one of a number of matrices often associated with a graph. We mention a few more.

**Definition 1.3.4.** *Let $G$ be an undirected graph with $n$ vertices $v_1, \dots, v_n$ and $m$ edges $e_1, \dots, e_m$.*

- *The* incidence matrix *of $G$ is the $n \times n$ matrix $C$ that has a 1 in position $(i, j)$ if the vertex $v_i$ is incident with the edge $e_j$, and zeros elsewhere.*

- *An* oriented incidence matrix *of $G$ is the $n \times m$ matrix $B$ defined by first assigning a direction to every edge of $G$ and then setting*
$$B_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the start vertex of } e_j \\ -1 & \text{if } v_i \text{ is the end vertex of } e_j \\ 0 & \text{otherwise} \end{cases}$$

The oriented incidence matrix depends on a choice of ordering of both the vertices and edges, and on a choice of orientation of the edges.

Given matrices that are associated with graphs, a general philosophy is to consider how the properties of the matrix and the graph are related to each other. In the case of an oriented incidence matrix, the rank of the matrix tells us about the number of connected components in the graph.

**Theorem 1.3.5.** *Let $G$ be a simple graph with $n$ vertices and $m$ edges, and let $B$ be an oriented incidence matrix of $G$. Then the rank of $B$ is $n - t$, where $t$ is the number of connected components of $G$.*

*Proof.* First suppose that G is connected. This means that for any pair of vertices $v_i$ and $v_j$ in G, there exists a walk in G from $v_i$ to $v_j$. We consider the left nullspace N of the matrix B. Note that every column of B has one entry equal to 1, one equal to $-1$, and is otherwise full of zeros. This means that the vector $(1\ 1\ \dots\ 1)$ belongs to the left nullspace of V, so this nullspace is at least 1-dimensional.

Suppose that $(a_1\ a_2 \dots a_n)$ is a non-zero element of N, and choose $k$ for which $a_k \neq 0$, write $a_k = \alpha$. Then $(a_1\ a_2 \dots a_n)$ must satisfy $(a_1\ a_2 \dots a_n)v = 0$ for every column $v$ of B, and in particular for those columns corresponding to edges that are incident with the vertex $v_k$. It follows that $a_i = a_k = \alpha$ whenever $v_i$ is adjacent to $v_k$. Now by the same reasoning applied to the neighbours of $v_k$, we must have $a_j = \alpha$ whenever $v_j$ is adjacent to a neighbour of $v_k$. Since G is connected, repetition of this step reaches all vertices of G and we conclude that $a_i = \alpha$ for all $i$ and that N is a 1-dimensional space. Thus $n = 1 + \text{rank}(B)$ and $\text{rank}(B) = n - 1$.

Now suppose that G has $t$ connected components and let their numbers of vertices be $n_1, n_2, \dots, n_t$, with $m_1, m_2, \dots, m_t$ edges respectively. By ordering the vertices component by component, we can arrange that B has a rectangular block diagonal structure with a $n_1 \times n_1$ block in the upper left, etc. Each block is an oriented incidence matrix of a connected component of G, and so a block with $n_i$ vertices has rank $n_i - 1$. It follows that the total rank is
$$(n_1 - 1) + (n_2 - 1) + \cdots + (n_t - 1) = n - t.$$
□

**Theorem 1.3.6.** *Let* $B$ *be an oriented incidence matrix of an undirected simple graph* $G$. *Then*

$$BB^T = D - A,$$

*where* $D$ *is the diagonal matrix whose diagonal entries are the total degrees of the vertices, and* $A$ *is the adjacency matrix of* $A$.

*Proof.* For $i = 1, \dots, n$, the entry in the $(i, i)$-position of $BB^T$ is just the ordinary scalar product of Row $i$ of $B$ with itself. Since every entry of this row is 1 or $-1$ or 0, this scalar product is the number of non-zero entries in Row $i$, which is the total degree of the vertex $v_i$.

Note that each Column of $B$ has exactly two non-zero entries, which are equal to 1 and $-1$. For $i \neq j$, the entry in the $(i, j)$ position of $BB^T$ is the scalar product of Row $i$ and Row $j$ of $B$. If this is not zero it means that there is a Column whose only two non-zero entries occur in positions $i$ and $j$, which means exactly that $v_i v_j$ is an edge in $G$. There can be at most one such column since there are no multiple edges in $G$. So the $(i, j)$ entry of $BB^T$ is $-1$ if $v_i$ and $v_j$ are adjacent in $G$ and by 0 otherwise. We conclude that $BB^T = D - A$. □

Note that a consequence of Theorem 1.3.6 is that the matrix $BB^T$ does not depend on the choice of orientation of the edges.

**Definition 1.3.7.** *Let* $G$ *be an undirected graph with adjacency matrix* $A$. *The matrix* $L = D - A$ *is called the Laplacian matrix of* $G$. *Its entries on the main diagonal are the degrees of the vertices of* $G$. *Away from the main diagonal, the entry in position* $(i, j)$ *is* $-1$ *or* 0 *according to whether* $v_i$ *and* $v_j$ *are adjacent or not.*

Properties of the Laplacian matrix of a graph G carry extensive information about properties of G itself. Moreover, as a real symmetric matrix it enjoys various special properties. For instance, it is a consequence of the following lemma that the rank of $L$ tells us the number of connected components of $G$.

**Lemma 1.3.8.** *Suppose that* $A \in M_{n \times p}(\mathbb{R})$. *Then the rank of the* $n \times n$ *matrix* $AA^T$ *is equal to the rank of* $A$.

*Proof.* That $\text{rank}(AA^T) \leqslant \text{rank}(A)$ is clear, since every columnn of $AA^T$ is a real linear combination of the columns of $A$. We show now that in this special case, the right nullspace of $AA^T$ is equal to the right nullspace of $A^T$. Suppose that $A^T v \neq 0$ for some $v \in \mathbb{R}^n$ Then $A^T v$ belongs to the columnspace of $A^T$, and since $A^T v$ is a non-zero vector in $\mathbb{R}^p$, it follows that $(A^T v)^T A^T v = v^T AA^T v \neq 0$. Thus $AA^T v \neq 0$, and $v$ does not belong to the right nullspace of $AA^T$. Then the right nullspaces of $A^T$ and $AA^T$ coincide and have the same dimension $d$, and the ranks of $AA^T$ and $A^T$ (and A) are all equal to $n - d$. □

As a real symmetric matrix, $L$ has the property that its eigenvalues are all real.

**Lemma 1.3.9.** *Let* $A$ *be a complex Hermitian* $n \times n$ *matrix and let* $\lambda$ *be a complex eigenvalue of* $A$. *Then* $\lambda \in \mathbb{R}$.

*Note* That $A$ is Hermitian means that $A = A^*$, where $A^*$ denotes the *Hermitian conjugate* of $A$, whose entries are the complex conjugates of the entries of $A^T$. A real symmetric matrix is a special case of a complex Hermitian matrix.

*Proof.* Since $A$ is Hermitian we have for any vector $v \in \mathbb{C}^n$ that $v^* A v \in \mathbb{R}$, since

$$(v^* A v)^* = v^* A^* v = v^* A v.$$

Thus $v^* A v$ is a complex number that is equal to its own complex conjugate, hence it is real.
Now let $u \in \mathbb{C}^n$ be an eigenvector corresponding to $\lambda$. Then

$$u^* A u \in \mathbb{R} \Longrightarrow u^* \lambda u \in \mathbb{R} \Longrightarrow \lambda u^* u \in \mathbb{R}.$$

Since $v^* v \in \mathbb{R}$ (since it is the sum of the entries of $u$ each multiplied by its own complex conjugate) it follows that $\lambda \in \mathbb{R}$ also. □

If G is a graph, it is a consequence of Lemma 1.3.9 that the eigenvalues of the Laplacian matrix L of G are real numbers. In fact they are all non-negative, for let $v$ be an eigenvector of L corresponding to the eigenvalue $\lambda$, and let B be an oriented incidence matrix of G. Then

$$v^T L v = v^T \lambda v = \lambda v^T v.$$

On the other hand

$$v^T L v = v^T B B^T v = (v^T B)(B^T v) = (B^T v)^T (B^T v).$$

Since $v^T v$ is a positive real number and $(B^T v)^T (B^T v)$ is a non-negative real number, it follows that $\lambda \geqslant 0$.

How the eigenvalues of L are related to properties of G is one of the themes of *spectral graph theory*. We will be able to prove the following statements.

1. We know that 0 occurs at least once as an eigenvalue of L. We will show that it occurs exactly once if and only if G is connected.

2. If G is connected, let $\mu$ be the least positive eigenvalue of L. This number is called the *algebraic connectivity* of G. We will show that it can be considered as a measure of how robustly connected the graph is. It is bounded above by the *vertex connectivity*, which is the least number of vertices whose removal disconects G.

3. The determinant of any $(n-1) \times (n-1)$ submatrix of L is the number of *spanning trees* in G. A subgraph of G is a spanning tree if it involves all the vertices of G, is connected, and has no cycles.

We will return to these statements later after some investigations of determinants and eigenvalues, in general and for the special case of symmetric matrices.