MA500-1: ADVANCED LINEAR ALGEBRA
SEMESTER 1 2015-16
PROBLEM SHEET 3
Due date: Wednesday November 25 2015

1. Let A and B be 2×2 matrices with entries in a field $\mathbb{F}$. Prove that $\det(AB) = \det(A)\det(B)$.

2. Let A and B be $n \times n$ matrices with entries in a field $\mathbb{F}$. Prove that $\det(AB) = \det(A)\det(B)$. This is difficult, so it is broken into the following steps.

   (a) Show that
   $$\det(AB) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \left(\sum_{k=1}^{n} A_{1k} B_{k\sigma(1)} \right) \left(\sum_{k=1}^{n} A_{2k} B_{k\sigma(2)} \right) \dots \left(\sum_{k=1}^{n} A_{nk} B_{k\sigma(n)} \right).$$

   (b) Let T_n denote the set of all functions from the set $\{1, 2, \dots, n\}$ to itself. (So T_n contains S_n but not all elements of T_n are permutations). Adapt part (a) above to show that
   $$\det(AB) = \sum_{\theta \in T_n} \left(A_{1\theta(1)} A_{2\theta(2)} \dots A_{n\theta(n)} \times \sum_{\sigma \in S_n} \text{sign}(\sigma) B_{\theta(1)\sigma(1)} B_{\theta(2)\sigma(2)} \dots B_{\theta(n)\sigma(n)} \right).$$

   (c) For every $\theta \in T_n$, show that
   $$B_{\theta(1)\sigma(1)} B_{\theta(2)\sigma(2)} \dots B_{\theta(n)\sigma(n)}$$
   is the determinant of the matrix B'_θ obtained from B and θ as follows: for $i = 1, \dots, n$, Row i of B' is Row $\theta(i)$ of B.

   (d) Conclude from part (c) that $\det B'_\theta = 0$ if $\theta \notin S_n$, and that $\det B'_\theta = \text{sign}(\theta) \det B$ if $\theta \in S_n$. Deduce that $\det(AB) = \det(A)\det(B)$.

Definition. *Let A and B be matrices in $M_n(\mathbb{F})$. Then A and B are said to be* simultaneously diagonalizable *in $M_n(\mathbb{F})$ if there exists an invertible matrix $Q \in GL(n, \mathbb{F})$ for which $Q^{-1}AQ$ and $Q^{-1}BQ$ are both diagonal.*

3. Let A and B be simultaneously diagonalizable matrices in $M_n(\mathbb{F})$. Prove that $AB = BA$.

4. Suppose that A has distinct eigenvalues $\lambda_1, \dots, \lambda_n$ in $\mathbb{F}$, and that $AB = BA$.

   (a) Suppose that $v \in \mathbb{F}^n$ is an eigenvector of A. Use the facts that $AB = BA$ and that A has distinct eigenvalues to prove that v is also an eigenvector of B.

   (b) Deduce that A and B are simultaneously diagonalizable in $M_n(\mathbb{F})$.

5. (a) Let $A \in M_n(\mathbb{F})$ and suppose that A has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ in $\mathbb{F}$ (not necessarily distinct). Prove that A is *triangularizable* over $\mathbb{F}$, i.e. that there exists an invertible matrix $P \in M_n(\mathbb{F})$ for which $P^{-1}AP$ is upper triangular.

   (b) Give an example of a matrix in $M_2(\mathbb{R})$ that is not triangularizable.

6. (a) Suppose that $AB = BA$ and that both A and B have all their eigenvalues in the field $\mathbb{F}$. Prove that A and B are *simultaneously triangularizable* over $\mathbb{F}$, i.e. that there exists an invertible $P \in M_n(\mathbb{F})$ for which the matrices $P^{-1}AP$ and $P^{-1}BP$ are both upper triangular.

   (b) Show that matrices A and B that are simultaneously triangularizable need not necessarily satisfy $AB = BA$.

Comments overleaf

These problems are hard (in my opinion). That is not meant to be discouraging, but the opposite - don't expect to make immediate progress on all of them, and be encouraged when you do make some progress! You have all the knowledge needed to solve them, but organising that knowledge and applying it to these problems will probably take some persistent and careful thinking. It might help to discuss them together. These comments are intended to provide some guidance (hopefully).

1. This is a warm-up, it can be done by direct means since the formulae involved are manageable.

2. I don't think this one involves any really difficult concepts, it is difficult just because of the complexity of the formula involved and all the details that you have to keep track of. Part (a) is a direct application of the determinant formula to the matrix AB. It expresses $\det(AB)$ as the sum over all permuations of the (signed) products of n entries of AB, each of which is itself the sum of n terms involving an entry of A and an entry of B. To get from (a) to (b), look at the product of the n sums in the expression in (a), and think about what it will look like when it is expanded. It will be a sum of products each involving n entries from different rows of A and n entries from B. For $\theta \in T_n$, think about the terms involving $A_{1\theta(1)}A_{2\theta(2)}\dots A_n\theta_n$ - what does this term get multiplied by? You can have a go at (c) even before finishing (b) if you like. The point here is that the last part of the expression in (b) can be interpreted as the determinant of a matrix related to B. This determinant can be non-zero only if θ is a permutation, so only those θ that belong to S_n actually contribute to the formula in (b). For (d) note that if $\theta \in S_n$, then $\sum_{\sigma\in S_n} \mathrm{sign}(\sigma)B_{\theta(1)\sigma(1)}B_{\theta(2)\sigma(2)}\dots B_{\theta(n)\sigma(n)}$ is the determinant of a matrix that is obtained from B by permuting its rows according to σ. The determinant of this matrix is related to $\det(B)$ by the fact that swapping two rows changes the sign of the determinant. To figure out exactly what this means for the formula in (b) is the last piece of the puzzle. If all of this seems very daunting, it might be a good idea to figure it our first for the 3×3 case.

3. and 4. This is one of my favourite theorems about matrices. If two matrices are simulataneously diagonalizable, it means that there is a basis $\{v_1, \dots, v_n\}$ of $\mathbb{F}^n$ with the property that each v_i is an eigenvector of both A and B, which is quite a special thing. For 3., use the fact that diagonal matrices commute with each other. The key part of all of this is 4(a). The fact that the eigenvalues of A are distinct means that each of them has only a 1-dimensional space of eigenvalues. For each i, let v_i be an eigenvector of A corresponding to λ_i. Since the λ_i are distinct, we know that $\{v_1, \dots, v_n\}$ is a basis of $\mathbb{F}^n$ (this was Theorem 1.1.14 in our lecture notes). Now look at BAv_1. This is λBv_1, since $Av_1 = \lambda_1 v_1$. Then, since $BA = AB$, we have $ABv_1 = \lambda B_{v_1}$. What does this tell us about v_1 and B? Remember that what we want to show is that each v_i must be an eigenvector of B.

5. You can do this by induction on n. The following is an outline. Let v_1 be an eigenvector of A corresponding to v_1. Choose a basis of $\mathbb{F}^n$ whose first element is v_1. Then rewriting with respect to this basis, we see that A is similar to a matrix whose first column has entries $\lambda_1, 0, 0, \dots, 0$, whose first row can have any entries, and which has a $(n-1) \times (n-1)$ matrix A_1 in the lower right region. By similarity, the eigenvalues of A_1 are $\lambda_2, \dots, \lambda_n$, and A_1 is triangularizable by the induction hypothesis.
For part (b), what you need is a matrix that has no real eigenvalue. What kind of familiar linear transformation fixes no line in $\mathbb{R}^2$?

6. For (a), we can use an induction argument as in Question 5, but we need to know first that A and B have a simultaneous eigenvector. Let E_1 be the eigenspace of A corresponding to λ_1 (we do not know that E_1 is 1-dimensional, that is the difference between this question and Q4). For any $v \in E_1$, show (as in Q4), that $Bv \in E_1$ also. Since all eigenvalues of B are in $\mathbb{F}$, this means that there is an element of E_1 that is an eigenvector of B as well as A. This is what is needed for the induction step.