

MA500-1: ADVANCED LINEAR ALGEBRA  
SEMESTER 1 2015-16  
PROBLEM SHEET 2  
Due date: Wednesday October 28 2015

1. Suppose that  $P$  is an invertible matrix in  $M_n(\mathbb{F})$  and that  $A$  is a matrix with  $n$  rows. Prove that  $\text{rank}(PA) = \text{rank}(A)$ .  
If  $\text{rank}(PA) = \text{rank}(A)$ , must it follow that  $P$  is invertible?

2. For  $p \times n$  matrices  $A$  and  $B$ , prove that

$$|\text{rank}(A) - \text{rank}(B)| \leq \text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

3. For matrices  $A \in M_{p \times q}(\mathbb{F})$  and  $B \in M_{q \times n}(\mathbb{F})$ , prove that

$$\text{rank}(A) + \text{rank}(B) - q \leq \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}.$$

4. Let  $A \in M_n(\mathbb{F})$  (so  $A$  is a square matrix). For a positive integer  $i$ , let  $r_i$  denote the rank of the matrix  $A^i$ .

(a) Prove that  $r_1 \geq r_2 \geq r_3 \geq \dots$

(b) Since the sequence  $r_1, r_2, \dots$  is a nonincreasing sequence of nonnegative integers, it must stabilize at some point. Let  $k$  be the least integer for which  $r_k = r_{k+1}$ . Show that  $r_j = r_k$  for all  $j \geq k$ , and deduce that  $k \leq n$ .

(c) For  $i \in \mathbb{N}$ , define  $s_i = \text{rank}(A^i) - \text{rank}(A^{i+1})$ . Prove that  $s_1 \geq s_2 \geq \dots$

5. A square matrix  $A$  is called *nilpotent* if  $A^k = 0$  for some positive integer  $k$ .

(a) If  $A$  is a nilpotent  $n \times n$  matrix, use Problem 4 to prove that  $A^n = 0$ .

(b) Prove that every strictly upper triangular matrix is nilpotent.

(c) Prove that every nilpotent matrix is similar to a strictly upper triangular matrix.

6. A bilinear form  $\tau$  on  $M_n(\mathbb{F})$  is defined by  $\tau(A, B) = \text{trace}(A^T B)$ .

(a) Show that  $\tau$  is symmetric.

(b) Show that  $\tau$  is nondegenerate.

(c) Identify the orthogonal complement with respect to  $\tau$  of the space  $S_n(\mathbb{F})$  of *symmetric* matrices in  $M_n(\mathbb{F})$ .

(d) Identify the orthogonal complement with respect to  $\tau$  of the space  $UT_n(\mathbb{F})$  of *upper triangular* matrices in  $M_n(\mathbb{F})$ .

7. Suppose that  $\tau$  is a nondegenerate alternating bilinear form on a finite dimensional  $\mathbb{F}$ -vector space  $V$ . Prove that there exists a basis of  $V$  with respect to which the Gram matrix of  $\tau$  has the form

$$\begin{pmatrix} 0_{k \times k} & I_k \\ -I_k & 0_{k \times k} \end{pmatrix},$$

where  $2k$  is the dimension of  $V$ .

8. A graph  $G$  with vertex set  $V$  is *bipartite* if  $V$  is the disjoint union of subsets  $V_1$  and  $V_2$ , and every edge of  $G$  joins a vertex of  $V_1$  to a vertex of  $V_2$  (so there are no edges within  $V_1$  or  $V_2$ ). Suppose that  $G$  is a bipartite graph, and let  $A$  be the adjacency matrix of  $G$ . Prove that  $\text{trace}(A^3) = 0$ .

## REMARKS ON THE PROBLEMS

Many of these problems are challenging, but you have four weeks to think about them, which will hopefully be enough time to get used to the kind of thinking involved. This page has a few general comments and some ideas that might help you to get started. Maybe it's good to have a go at the problems first without reading the comments though. For some of them there are lots of good approaches that don't necessarily follow the lines suggested in these remarks.

1. Note that every row of  $PA$  is a linear combination of the rows of  $A$ . What does this tell you about the relationship between the ranks of  $PA$  and  $A$ ? Now what about  $P^{-1}(PA)$ ?
2. There are two parts here. The easier one (I think) is to show that  $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$ . To do this, think about the rows of  $A+B$ . Each of them is the sum of a row of  $A$  and a row of  $B$ . So you can use a basis for the row space of  $A$  and a basis for the row space of  $B$  to get an upper bound for the dimension of the row space of  $A+B$ .  
For the other inequality, first assume that  $\text{rank}(A) \geq \text{rank}(B)$ . Then you need to prove that  $\text{rank}(A) - \text{rank}(B) \leq \text{rank}(A+B)$ . Can you rewrite this in order to obtain it as a consequence of the first part of the problem?
3. Again there are two things to do here. The easier one is to show that  $\text{rank}(AB) \leq \text{rank}(A)$  and  $\text{rank}(AB) \leq \text{rank}(B)$ .  
For the other inequality, think of  $B$  as a linear transformation from  $\mathbb{F}^n$  to  $\mathbb{F}^q$ , and of  $A$  as a linear transformation from  $\mathbb{F}^q$  to  $\mathbb{F}^p$ . Then  $AB$  represents the composition of the two, which goes from  $\mathbb{F}^n$  to  $\mathbb{F}^p$ . You need to figure out the minimum possible dimension of the image of this mapping. Bear in mind that the dimension of the kernel of the transformation described by  $A$  is  $q - \text{rank}(A)$ , how this intersects the image of  $B$  is a key consideration.
4. Here it may help to think of  $A$  as a linear transformation that is repeatedly being composed with itself. Remember that every column of  $A^{i+1}$  is a linear combination of the columns of  $A^i$ . If  $\text{rank}(A^i) = \text{rank}(A^{i+1})$ , it means that  $A^i$  and  $A^{i+1}$  have the same image, and that this image has trivial intersection with the kernel of  $A$  (think about this).
5. Part (c) is the tricky one here. Let  $A$  be a nilpotent  $n \times n$  matrix, and think of  $A$  as a linear transformation  $f$  from  $\mathbb{F}^n$  to  $\mathbb{F}^n$ . Build up a sequence of subspaces of  $\mathbb{F}^n$  as follows. Let  $U_1 = \ker f$ . Let  $U_2 = \{x \in \mathbb{F}^n : f(x) \in U_1\}$ , let  $U_3 = \{x \in \mathbb{F}^n : f(x) \in U_2\}$ , and so on. Then  $U_1 \subseteq U_2 \subseteq U_3 \dots$ , and the nilpotency of  $f$  means that  $U_k = \mathbb{F}^n$  for some  $k$ . Now build up a basis of  $\mathbb{F}^n$  as follows: start with a basis of  $U_1$ , extend it to a basis of  $U_2$ , then  $U_3$  and so on. Show that the matrix of  $f$  with respect to this basis is strictly upper triangular.
7. Start with any non-zero element  $b_1$  of  $V$ . Since  $\tau$  is nondegenerate there exists  $b_{k+1} \in V$  with  $\tau(b_1, b_{k+1}) = 1$ . Now  $\{b_1, b_{k+1}\}$  is a linearly independent set (why)? Write  $U_1 = \langle b_1, b_{k+1} \rangle$ . Now move to  $U_1^\perp$  (which has trivial intersection with  $U_1$ ) and extend the basis there.
8. What does the trace of  $A^3$  count?