Character Theory of Finite Groups 2005-2006

Problem Sheet 3

1. Let G be a finite group. A *class function* on G is a function $\phi : G \longrightarrow \mathbb{C}$ with the property that $\phi(x) = \phi(y)$ whenever x and y are conjugate elements of G.

   (a) Show that the set S of class functions on G is a vector space over C, whose dimension is the number of conjugacy classes in G.

   (b) Show that the set of irreducible complex characters of G is a basis for this vector space.

2. Let G be a finite group. Prove that the elements x and y of G are conjugate to each other in G if and only if $\chi(x) = \chi(y)$ for every irreducible complex character χ of G.

3. Prove that every finite group of order three or greater has at least three irreducible complex characters.

4. Let G be a finite group. Show that the number of irreducible characters of degree 1 of G is $[G : G']$, the index in G of its commutator subgroup.

5. Let G be a finite group with irreducible complex characters $\chi_1, \ldots, \chi_k$, and let $g \in G$. Prove that

$$\sum_{i=1}^{k} \chi(g)\chi(g^{-1}) = |G|$$

   if and only if $g \in \mathcal{Z}(G)$.

6. Let A be an abelian subgroup of the finite group G, and suppose that G has an irreducible complex character χ for which $\chi(1) = [G : A]$. Prove that $\chi(g) = 0$ for all those elements g of G that do not belong to A.

7. Let $R : G \longrightarrow GL(m, \mathbb{C})$ be a non-trivial irreducible representation of a finite group G. Prove that

$$\sum_{g \in G} R(g)$$

   is the zero matrix.

8. Let G be a finite group with conjugacy classes $C_1, \ldots, C_k$, and for $i = 1, \ldots, k$ let g_i be an element of C_i. Let C denote the character table of G, regarded as a $k \times k$ matrix over $\mathbb{C}$. Prove that

$$\overline{\det C} \det C = \prod_{i=1}^{k} |C_G(g_i)|.$$

   (The bar denotes complex conjugation).

Comments and Hints overleaf

Remarks on the Problems

1. To show that $\dim S$ is the number of conjugacy classes in G, write down a basis of S in which the basis elements correspond to the different conjugacy classes. Part (b) follows easily from part (a) and Lemma 3.2.1.

2. If x is not conjugate to y, there certainly exists a class function ϕ on G for which $\phi(x) \neq \phi(y)$. Use Problem 1.

3. If a group $|G|$ of order $n \geq 3$ has only two irreducible complex characters, then all the non-identity elements of G must belong to the same conjugacy class. Why can't this happen?

4. The factor group G/G' is abelian and its order is $[G : G']$. Thus every irreducible complex character of G/G' has degree 1, and the number of them is $[G : G']$. If χ is a character of G of degree 1, show that $G' subseteq \ker \chi$. This means that χ is constant on cosets of G' in G and that the function $\chi' : G/G' \longrightarrow \mathbb{C}$ defined by $\chi'(xG') = \chi(x)$ is a character of G/G'. Show that the $\chi \longleftrightarrow \chi'$ is a bijective correspondence between the set of irreducible characters of degree 1 of G and the set of irreducible characters of G/G'.

5. Use the column orthogonality relation.

6. Use the row orthogonality relation.

7. Let $A = \sum_{g \in G} R(g)$ and let V denote the vector space of column vectors of length m over $\mathbb{C}$. Since R is irreducible V contains no proper subspace V_1 with the property that $R(g)v \in V_1$ for all $g \in G$ and $v \in V_1$. Let v_1 denote the first column of A regarded as an element of V. Show that $R(g)v_1 = v_1$ for every element g of G. Conclude that v_1 is the zero element of V, using the irreducibility of R. Repeat this argument to show that $A = 0$.

8. Let $\bar{C}^t$ denote the matrix obtained from C by first transposing and then replacing each entry with its complex conjugate. Consider the matrix product $\bar{C}^t C$. What are the entries of this $k \times k$ matrix? They can be described using the column orthogonality relations.