Recall from Week 8

G acts on a set S, means that each element g of G permutes the elements of S, taking x to $g \cdot x$.

- $g \cdot (h \cdot x) = gh \cdot x$, for all $g, h \in G$ and all $x \in S$.
- $\mathrm{id}_G \cdot x = x$ for all $x \in S$.

The orbit of an element $x \in S$ is $G \cdot x = \{g \cdot x : g \in G\} \subseteq S$.
For $x, y \in S$

- $x \in G \cdot x$ since $x = \mathrm{id}_G \cdot x$.
- $G \cdot x = G \cdot y$ if $y \in G \cdot x$ (then $y = h \cdot x$ and $x = h^{-1} \cdot y$)
- $G \cdot x \cap G \cdot y = \emptyset$ otherwise.

S is the disjoint union of distinct orbits under the action of G.

The Stabilizer of an Element

Let $x \in S$. The stabilizer of x, denoted $\mathrm{Stab}_G(x)$, is the set of elements g of G that fix x.

$$\mathrm{Stab}_G(x) = \{g \in G : g \cdot x = x\} \subseteq G.$$

Lemma $\mathrm{Stab}_G(x)$ is a subgroup of G.

Proof

1. $\mathrm{id}_G \in \mathrm{Stab}_G(x)$, since $\mathrm{id}_G \cdot x = x$.
2. Closure: suppose $g, h, \in \mathrm{Stab}_G(x)$. We show $gh \in \mathrm{Stab}_G(x)$.

$$gh \cdot x = g \cdot (h \cdot x) = g \cdot x = x \Longrightarrow gh \in \mathrm{Stab}_G(x).$$

3. Inverse: Suppose $g \in \mathrm{Stab}_G(x)$. We show $g^{-1} \in \mathrm{Stab}_G(x)$.

$$\begin{aligned} g \cdot x = x &\implies g^{-1} \cdot (g \cdot x) = g^{-1} \cdot x \\ &\implies \mathrm{id}_G \cdot x = g^{-1} \cdot x \\ &\implies x = g^{-1} \cdot x. \end{aligned}$$

The Orbit-Stabilizer Theorem

For a finite group G acting on a set S, and for any $x \in S$, the number of elements in the orbit of x is the index in G of the stabilizer of x.

$$|G \cdot x| = [G : \mathrm{Stab}_G(x)]$$

Proof For elements g and h of G, we consider when $g \cdot x$ and $h \cdot x$ are the same (or different) elements of S.

$$\begin{aligned}
g \cdot x = h \cdot x &\iff g^{-1} \cdot (g \cdot x) = g^{-1} \cdot (h \cdot x) \\
&\iff \mathrm{id}_G \cdot x = g^{-1} h \cdot x \\
&\iff x = g^{-1} h \cdot x \\
&\iff g^{-1} h \in \mathrm{Stab}_G(x) \\
&\iff h \in g \, \mathrm{Stab}_G(x).
\end{aligned}$$

So $g \cdot x = h \cdot x$ if and only if g and h are in the same left coset of $\mathrm{Stab}_G(x)$ in G. So the number of distinct elements of $G \cdot x$ is the number of distinct left cosets of $\mathrm{Stab}_G(x)$ in G.

Special Case: the conjugation action

Every group G acts on the set of its own elements via the conjugation action, defined for $g, x \in G$ by

$$g \cdot x = gxg^{-1}.$$

Then

- $id_G \cdot x = x$ for all $x \in G$.
- $g \cdot (h \cdot x) = g \cdot (hxh^{-1}) = ghxh^{-1}g^{-1} = ghx(gh)^{-1} = gh \cdot x$.

Under this action, the orbit of $x \in G$ is the conjugacy class of x. The stabilizer of x is the centralizer of x in G. So the Orbit-Stabilizer Theorem tells us that the number of distinct conjugates of x in G is $[G : C_G(x)]$. We already knew this (Theorem 2.2.10).

Week 9, Challenge 1

A group of order 45 acts on a set with 16 elements. Prove that there is an element in the set whose stabilizer is the entire group.