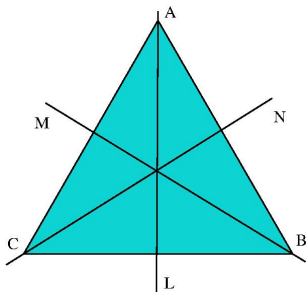


Cayley's Theorem



\circ	id	R_{120}	R_{240}	T_L	T_M	T_N
id	id	R_{120}	R_{240}	T_L	T_M	T_N
R_{120}	R_{120}	R_{240}	id	T_M	T_N	T_L
R_{240}	R_{240}	id	R_{120}	T_N	T_L	T_M
T_L	T_L	T_N	T_M	id	R_{240}	R_{120}
T_M	T_M	T_L	T_N	R_{120}	id	R_{240}
T_N	T_N	T_M	T_L	R_{240}	R_{120}	id

Each row in the table for D_6 details the permutation of the six elements that results from composing everything on the left with the element that labels the row. So we associate to each element of D_6 a different permutation of six objects (which happen to be the six elements of D_6). By taking this view we can interpret D_6 as a subgroup of S_6 . This is [Cayley's Theorem](#).

Isomorphism

An **isomorphism** between groups (G, \star_G) and (H, \star_H) is a bijective function $\phi : G \rightarrow H$ with the property that

$$\phi(x \star_G y) = \phi(x) \star_H \phi(y), \text{ for all } x, y \in G.$$

This means that ϕ matches the elements of G with those of H , in a way that matches the group operations too. The groups become identical after relabelling their elements according to the matching.

Example The group of complex 4th roots of unity under multiplication is isomorphic to group of integers modulo 4 under addition.

(G_4, \times)	1	i	-1	$-i$	$(\mathbb{Z}_4, +)$	$[0]_4$	$[1]_4$	$[2]_4$	$[3]_4$
1	1	i	-1	$-i$	$[0]_4$	$[0]_4$	$[1]_4$	$[2]_4$	$[3]_4$
i	i	-1	$-i$	1	$[1]_4$	$[1]_4$	$[2]_4$	$[3]_4$	$[0]_4$
-1	-1	$-i$	1	i	$[2]_4$	$[2]_4$	$[3]_4$	$[0]_4$	$[1]_4$
$-i$	$-i$	1	i	-1	$[3]_4$	$[3]_4$	$[0]_4$	$[1]_4$	$[2]_4$

Cayley's Theorem - statement and outline proof

Theorem (Cayley, 1854) Let G be a group of order n . Then G is isomorphic to a subgroup of S_n .

Outline proof G acts on the set of its own elements by left multiplication. For $g \in G$, we write ϕ_g for the permutation of the elements of G defined by left multiplication by g .

$$\phi_g(x) = gx, \text{ for } x \in G.$$

If $h \in G$ and $h \neq g$, then $\phi_h \neq \phi_g$.

The mapping $g \rightarrow \phi_g$ associates to every element of G a permutation of the n elements of G , and these permutations satisfy

$$\phi_{gh}(x) = ghx = g(hx) = \phi_g\phi_h(x), \text{ for all } g, h, x \in G.$$

Moreover ϕ_{id} is the identity permutation and $\phi_{g^{-1}}$ is the inverse of ϕ_g , so the set of all ϕ_g , where $g \in G$, is a group of permutations of n objects that is isomorphic to G .

Week 9, Challenge 2

Cayley's Theorem tells us that D_8 (the group of symmetries of the square) is isomorphic to a subgroup of S_8 . The order of S_8 is $8! = 40,320$, so D_8 is a really tiny subgroup of it. Cayley's Theorem recognizes that D_8 is (isomorphic to) a subgroup of S_8 , by considering each element of D_8 as a permutation of the eight elements of D_8 . By considering actions of D_8 as permutations of sets with fewer than 8 elements, find the least k for which D_8 is isomorphic to a subgroup of S_k . Give an explanation to show that your k is the least possible.