Week 8 Challenge

Let $G = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R}, a^2 + b^2 = 1\}$
Show that G is a group under matrix multiplication.

1. Identity: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G$

2. Closure: $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ $B = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$

$A \star B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{pmatrix}$

$(ac - bd)^2 + (ad + bc)^2 = a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2$

$= (a^2 + b^2)(c^2 + d^2) = 1 \implies A \star B \in G$

3. Inverse: $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ $A^{-1} = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}$

where $c = -b, c \in \mathbb{R}, a^2 + b^2 = a^2 + c^2 = 1 \implies A^{-1} \in G$

4. Associativity: Matrix multiplication is associative.

$\implies$ G is a group under matrix multiplication

G acts on $\mathbb{R}$ as a group of invertible linear transformations (or by matrix-vector multiplication if we interpret $\mathbb{R}^2$ as the column vector of length 2). What are these transformations geometrically? Give a geometric description of the orbits of this action.

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix}$$

Geometrically these transformations are rotations about the origin through θ. Where $a = cos\theta$ and $b = sin\theta$. Therefore, $a, b \in \mathbb{R}$ and $a^2 + b^2 = cos^2\theta + sin^2\theta = 1$.

The orbit of $(x, y) \in \mathbb{R}^2$ is the set of all points on the circle with radius equal to the Euclidean distance from the origin to (x, y).
$G \cdot (x, y) = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 = x^2 + y^2\}$