Definition of a Group Action

Definition Let G be a group and let S be a set. We say that G acts on S if every element g of G corresponds to a permutation π_g of S, satisfying the following two conditions:

1. π_{id} is the identity permutation of S.
2. If g and h are elements of G, then $\pi_{gh} = \pi_g \circ \pi_h$.

Notation If $g \in G$ and $x \in S$, it is conventional to write $g \cdot x$ for the element of S obtained by applying the permutation determined by g to x. In this notation, the above conditions are

1. $\mathrm{id} \cdot x = x, \ \forall x \in S$.
2. $g \cdot (h \cdot x) = (gh) \cdot x, \forall x \in S, \ \forall g, h \in G$.

Orbits and Stabilizers

Let G be a group acting on a set S.

Definition For $x \in S$, the orbit of x, denoted $G \cdot x$, is the set of elements of S that are obtained by applying an element of G to x.

$$G \cdot x = \{g \cdot x : g \in G\}. \subseteq S$$

a subset of S, note $x \in G \cdot x$, since $id \cdot x = x$

Definition For $x \in S$, the stabilizer of x in G, denoted $\boxed{\mathrm{Stab}_G(x)}$, is the subset of G consisting of those elements that fix x.

$$\mathrm{Stab}_G(x) = \{g \in G : \; g \cdot x = x\}.$$

$\mathrm{Stab}_G(x) \subseteq G$

Note $id \in \mathrm{Stab}_G(x)$ for all $x \in S$, so $\mathrm{Stab}_G(x)$ is not empty

Two Lemmas

1. For $x, y \in \cancel{G}$ S, $G \cdot x$ and $G \cdot y$ are either equal or disjoint, and S is the disjoint union of the distinct orbits under the action of G
2. For $x \in S$, $\mathrm{Stab}_G(x)$ is a subgroup of G.

(1) If x, y are elements of S, the the orbits $G \cdot x$ and $G \cdot y$ are identical or disjoint.

- If they are disjoint, nothing to do.
- If they are not disjoint, then there are elements g and h of G for which

$$g \cdot x = h \cdot y \quad \text{in } S$$

This implies $\underbrace{g^{-1} \cdot (g \cdot x)} = \underbrace{g^{-1} \cdot (h \cdot y)}$

$$\Rightarrow \quad (g^{-1}g) \cdot x = id_G \cdot x = x = (g^{-1}h) \cdot y$$

$$\Rightarrow \quad x = (g^{-1}h) \cdot y$$

$$\Rightarrow \quad x \in G \cdot y$$

$$\Rightarrow \quad x = g' \cdot y \text{ for some } g' \in G$$

Then every element of $G \cdot x$ has the form

$$h' \cdot x \quad \text{or} \quad h' \cdot (g' \cdot y) = (h'g') \cdot y \text{ for some } h' \in G$$

This means $G \cdot x \subseteq G \cdot y$, similarly $G \cdot y \subseteq G \cdot x$

(2) $\mathrm{Stab}_G(x)$ is a subgroup of G

1. $id_G \cdot x = x$ by definition
$\Rightarrow id_G \in \mathrm{Stab}(x)$

2. Closure: Suppose $g, h \in \mathrm{Stab}_G(x)$ $g \in \mathrm{Stab}_G(x)$
Then $(gh) \cdot x = g(\underbrace{h \cdot x}_{x,\text{ since } h \in \mathrm{Stab}_G(x)}) = g \cdot x = x$

$\Rightarrow (gh) \cdot x = x \Rightarrow gh \in \mathrm{Stab}_G(x)$

3. Suppose $g \in \mathrm{Stab}_G(x)$. Need to show $g^{-1} \in \mathrm{Stab}_G(x)$ also.

$$g \cdot x = x$$

$$\Rightarrow g^{-1} \cdot (g \cdot x) = g^{-1} \cdot x$$

$$\Rightarrow id_G \cdot x = x = g^{-1} \cdot x \Rightarrow g^{-1} \in \mathrm{Stab}_G(x)$$

Challenge for Week 8

Let $G = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : \ a, b \in \mathbb{R}, \ a^2 + b^2 = 1 \right\}$.

- Show that G is a group under matrix multiplication.
- G acts on $\mathbb{R}^2$ as a group of invertible linear transformations (or by matrix-vector multiplication if we interpret $\mathbb{R}^2$ as the set of column vectors of length 2). What are these transformations, geometrically? Give a geometric description of the orbits of this action.