Definition of a Group Action

Definition Let G be a group and let S be a set. We say that G acts on S if every element g of G corresponds to a permutation π_g of S, satisfying the following two conditions:

1. π_{id} is the identity permutation of S.
2. If g and h are elements of G, then $\pi_{gh} = \pi_g \circ \pi_h$.

Notation If $g \in G$ and $x \in S$, it is conventional to write $g \cdot x$ for the element of S obtained by applying the permutation determined by g to x. In this notation, the above conditions are

1. $\mathrm{id} \cdot x = x, \ \forall x \in S$.
2. $g \cdot (h \cdot x) = (gh) \cdot x, \forall x \in S, \ \forall g, h \in G$.

Orbits and Stabilizers

Let G be a group acting on a set S.

Definition For $x \in S$, the orbit of x, denoted $G \cdot x$, is the set of elements of S that are obtained by applying an element of G to x.

$$G \cdot x = \{g \cdot x : g \in G\}.$$

Definition For $x \in S$, the stabilizer of x in G, denoted $\mathrm{Stab}_G(x)$, is the subset of G consisting of those elements that fix x.

$$\mathrm{Stab}_G(x) = \{g \in G : \; g \cdot x = x\}.$$

Two Lemmas

1. For $x, y \in G$, $G \cdot x$ and $G \cdot y$ are either equal or disjoint, and S is the disjoint union of the distinct orbits under the action of G
2. For $x \in S$, $\mathrm{Stab}_G(x)$ is a subgroup of G.

Challenge for Week 8

Let $G = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : \; a, b \in \mathbb{R}, \; a^2 + b^2 = 1 \right\}$.

- Show that G is a group under matrix multiplication.
- G acts on $\mathbb{R}^2$ as a group of invertible linear transformations (or by matrix-vector multiplication if we interpret $\mathbb{R}^2$ as the set of column vectors of length 2). What are these transformations, geometrically? Give a geometric description of the orbits of this action.