Conjugates of a single cycle

In S_7, calculate $\sigma\pi\sigma^{-1}$ (as a product of cycles), where

$$\pi = (1\ 2\ 3\ 4\ 5), \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 6 & 3 & 5 & 1 & 4 \end{pmatrix}$$

- The elements 1 and 4 are sent by σ^{-1} to 6 and 7, which are not moved by π, and then mapped respectively back to 1 and 4 by σ. So 1 and 4, which are the images under σ of the fixed points of π, are fixed points of $\sigma\pi\sigma^{-1}$.
- The elements σ^{-1} sends $2, 7, 6, 3, 5$ to $1, 2, 3, 4, 5$ respectively. Then π cycles these around, sending the list $1, 2, 3, 4, 5$ to $2, 3, 4, 5, 1$. Then σ maps the list $2, 3, 4, 5, 1$ back to $7, 6, 3, 5, 2$.
- Overall $\sigma\pi\sigma^{-1} = (2\ 7\ 6\ 3\ 5)$. In particular,$\sigma\pi\sigma^{-1}$ has the same cycle type as π, and the elements that it cycles are the images under σ of those cycled by π.

Conjugates of a general permutation

Suppose $\pi \in S_n$ and write $\pi = \pi_1 \pi_2, \ldots, \pi_k$ be the disjoint cycle representation of π. Let $\sigma \in S_n$. Then

$$\sigma\pi\sigma^{-1} = (\sigma\pi_1\sigma^{-1})(\sigma\pi_2\sigma^{-1})\ldots(\sigma\pi_k\sigma^{-1}).$$

This the disjoint cycle description of $\sigma\pi\sigma^{-1}$, since $(\sigma\pi_i\sigma^{-1})$ cycles the images under σ of the elements cycled by π_i.

Example In S_{10}, write

$$\pi = (1\ 4\ 8\ 7)(2\ 10\ 3\ 6\ 5),\ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 5 & 6 & 3 & 10 & 1 & 4 & 8 & 2 & 7 \end{pmatrix}$$

Then

$$(\sigma\pi\sigma^{-1}) = (9\ \ 3\ \ 8\ \ 4)(5\ \ 7\ \ 6\ \ 1\ \ 10).$$

In particular, elements of S_n that are conjugate to each other have the same cycle type, the same numbers of cycles of each length in their disjoint cycle description.

Same cycle type

Finally, if two elements of S_n do have the same cycle type, they are conjugate in S_n.
In S_8, write

$$\tau = (2\ 5\ 8)(1\ 6\ 7), \quad \pi = (4\ 1\ 6)(5\ 3\ 2).$$

Then τ and π have the same cycle type, and $\sigma\tau\sigma^{-1} = \pi$, provided that the permutation σ maps elements permuted by the 3-cycles of τ to elements permuted by the 3-cycles of π, preserving the cyclic order. For example we could take

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 7 & 8 & 1 & 3 & 2 & 6 \end{pmatrix}.$$

Partitions and Conjugacy Classes

So the conjugacy classes in S_n correspond to the possible cycle types of a permutation in S_n. The number of these is the number of ways to write n as a sum of positive integers: the partitions of n.

If $n = 7$:

- The partition $2 + 2 + 3$ corresponds to permutations with the same cycle type as

  $$(1\ 2)(3\ 4)(5\ 6\ 7).$$

  The number of these is $\binom{7}{3} \times 2! \times 3 = 210$.
- The partition $1 + 1 + 2 + 3$ corresponds to the permutations with the same cycle type as

  $$(1\ 2)(3\ 4\ 5).$$

  The number of these is $\binom{7}{3} \times 2! \times \binom{4}{2} = 420$.

Challenge for Week 7

In S_8, write

$$\tau = (2\ 5\ 8)(1\ 6\ 7), \quad \pi = (4\ 1\ 6)(5\ 3\ 2).$$

How many elements σ of S_8 have the property that $\sigma\tau\sigma^{-1} = \pi$?