Symmetric Groups

The group consisting of all permutations of a set of n elements, under the composition operation, is called the *symmetric group of degree n* and denoted S_n. The order of S_n is $n!$, and it is conventional to label the n objects being permuted as $1, 2, \ldots, n$.

An element of S_4 is a permutation of the set $\{1, 2, 3, 4\}$; this means a function from that set to itself that sends each element to a different image, and hence shuffles the four elements. In S_4, a basic way to represent the permutation $1 \to 1,\ 2 \to 4,\ 3 \to 2,\ 4 \to 3$ is by the array

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}.$$

The composition operation

In S_5, suppose that

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix}.$$

Calculate the products $\sigma\tau$ and $\tau\sigma$. Also, write down σ^{-1} in array format.

$\sigma\tau$ (σ after τ)

$$\begin{array}{ccccc} 1 & \xrightarrow{} & 4 & \xrightarrow{} & 4 \\ 2 & \xrightarrow{} & 2 & \xrightarrow{} & 3 \\ 3 & \xrightarrow{} & 3 & \xrightarrow{} & 5 \\ 4 & \xrightarrow{} & 5 & \xrightarrow{} & 1 \\ 5 & \xrightarrow{} & 1 & \xrightarrow{} & 2 \\ & \tau & & \sigma & \end{array}$$

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$$

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

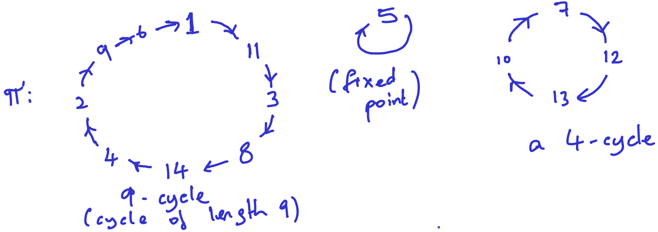
Note $\sigma\tau \neq \tau\sigma$ $\quad$ S_5 is not abelian

Disjoint cycle description

Look at the following permutation in S_{14}.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 11 & 9 & 8 & 2 & 5 & 1 & 12 & 14 & 6 & 7 & 3 & 13 & 10 & 4 \end{pmatrix}$$

Start with the element 1 and look at what happens to it when you repeatedly apply π.



Disjoint cycle representation

In that there were nine distinct elements in the sequence that started at 1. So the permutation π produces the following *cycle*:

$$1 \to 11 \to 3 \to 8 \to 14 \to 4 \to 2 \to 9 \to 6 \to 1$$

This cycle is often written using the following notation:

$$(1 \;\; 11 \;\; 3 \;\; 8 \;\; 14 \;\; 4 \;\; 2 \;\; 9 \;\; 6).$$

The set $\{1, 2, 3, 4, 6, 8, 9, 11, 14\}$ is called the orbit of 1 under π. We find two more cycles and two more orbits:

- (5), a fixed point
- $(7 \;\; 12 \;\; 13 \;\; 10)$, a cycle of length 4.

We write π as follows as a product of disjoint cycles:

$$\pi = (1 \;\; 11 \;\; 3 \;\; 8 \;\; 14 \;\; 4 \;\; 2 \;\; 9 \;\; 6)(7 \;\; 12 \;\; 13 \;\; 10).$$

Remarks on disjoint cycle representation

- Every permutation can be written as a product (composition) of disjoint cycles. The convention is to not include fixed points (cycles of length 1) in the written description, but you can if you like.
- Disjoint cycles are permutations in the relevant S_n. They commute with each other, because they shuffle separate sets of elements (that's what disjoint means).
- The written description of a permutation as a product of disjoint cycles is unique, except that
  - the order in which the different cycles are written can vary;
  - for each cycle, what matters is the cyclic order, not which element comes first or last, so the expressions $(1\ \ 2\ \ 3\ \ 4)$ and $(3\ \ 4\ \ 1\ \ 2)$ for example represent the same cycle.

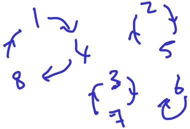
Order of an element in a symmetric group

Let G be a group and let $g \in G$. Recall that the order of the element g is the number of elements in the cyclic subgroup generated by g. If this is finite, it is the least k for which $g^k = \mathrm{id}_G$.

Question In S_8, what is the order of the element

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 7 & 8 & 2 & 6 & 3 & 1 \end{pmatrix}?$$

$(1\ \ 4\ \ 8)(2\ \ 5)(3\ \ 7)(6)$



Write $\pi = (1\ \ 4\ \ 8)(2\ \ 5)(3\ \ 7)$. Then $\pi^k = \mathrm{id}$ provided that k is a multiple of both 2 and 3.

The least such k is 6, this is the order of π in S_8.

The order of a permutation is the least common multiple of the cycle lengths in its description as a product of disjoint cycles.