

Symmetric Groups

The group consisting of all permutations of a set of n elements, under the composition operation, is called the *symmetric group of degree n* and denoted S_n . The order of S_n is $n!$, and it is conventional to label the n objects being permuted as $1, 2, \dots, n$.

An element of S_4 is a permutation of the set $\{1, 2, 3, 4\}$; this means a function from that set to itself that sends each element to a different image, and hence shuffles the four elements. In S_4 , a basic way to represent the permutation

$1 \rightarrow 1, 2 \rightarrow 4, 3 \rightarrow 2, 4 \rightarrow 3$ is by the array

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}.$$

The composition operation

In S_5 , suppose that

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix}.$$

Calculate the products $\sigma\tau$ and $\tau\sigma$. Also, write down σ^{-1} in array format.

Disjoint cycle description

Look at the following permutation in S_{14} .

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 11 & 9 & 8 & 2 & 5 & 1 & 12 & 14 & 6 & 7 & 3 & 13 & 10 & 4 \end{pmatrix}$$

Start with the element 1 and look at what happens to it when you repeatedly apply π .

Disjoint cycle representation

In that there were nine distinct elements in the sequence that started at 1. So the permutation π produces the following *cycle*:

$$1 \rightarrow 11 \rightarrow 3 \rightarrow 8 \rightarrow 14 \rightarrow 4 \rightarrow 2 \rightarrow 9 \rightarrow 6 \rightarrow 1$$

This cycle is often written using the following notation:

$$(1 \ 11 \ 3 \ 8 \ 14 \ 4 \ 2 \ 9 \ 6).$$

The set $\{1, 2, 3, 4, 6, 8, 9, 11, 14\}$ is called the **orbit** of 1 under π . We find two more cycles and two more orbits:

- ▶ (5), a fixed point
- ▶ (7 12 13 10), a cycle of length 4.

We write π as follows as a product of disjoint cycles:

$$\pi = (1 \ 11 \ 3 \ 8 \ 14 \ 4 \ 2 \ 9 \ 6)(7 \ 12 \ 13 \ 10).$$

Remarks on disjoint cycle representation

- ▶ Every permutation can be written as a product (composition) of disjoint cycles. The convention is to not include fixed points (cycles of length 1) in the written description, but you can if you like.
- ▶ Disjoint cycles are permutations in the relevant S_n . They commute with each other, because they shuffle separate sets of elements (that's what **disjoint** means).
- ▶ The written description of a permutation as a product of disjoint cycles is unique, except that
 - ▶ the order in which the different cycles are written can vary;
 - ▶ for each cycle, what matters is the cyclic order, not which element comes first or last, so the expressions $(1\ 2\ 3\ 4)$ and $(3\ 4\ 1\ 2)$ for example represent the same cycle.

Order of an element in a symmetric group

Let G be a group and let $g \in G$. Recall that the **order of the element g** is the number of elements in the cyclic subgroup generated by g . If this is finite, it is the least k for which $g^k = \text{id}_G$.

Question In S_8 , what is the order of the element

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 7 & 8 & 2 & 6 & 3 & 1 \end{pmatrix}?$$

Write $\pi = (1\ 4\ 8)(2\ 5)(3\ 7)$. Then $\pi^k = \text{id}$ provided that k is a multiple of both 2 and 3.

The least such k is 6, this is the order of π in S_8 .

The order of a permutation is the least common multiple of the cycle lengths in its description as a product of disjoint cycles.