The conjugates of an element

Let x be a (fixed) element of a group G, and let g be any element. We know that g commutes with x, or centralizes x, or belongs to the centralizer of x, if and only if

$$xg = gx \text{ in } G.$$

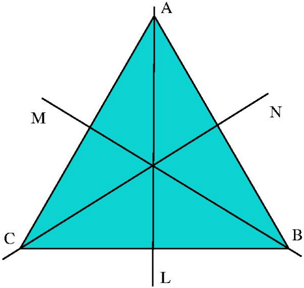
The equation $xg = gx$ can be rearranged (after composing on the right with g^{-1} on both sides) to

$$x = gxg^{-1}.$$

As g moves through the elements of G, the element gxg^{-1} is equal to x only when $g \in C_G(x)$. All of the elements that occur as gxg^{-1}, as g varies, are called the conjugates of x in G.

Context and Examples

1. Let A be a matrix in $\mathrm{GL}(3,\mathbb{R})$. The conjugates of A in $\mathrm{GL}(3,\mathbb{R})$ are all matrices of the form PAP^{-1}, where P is an invertible matrix. These are exactly the matrices that represent the same linear transformation as A, with respect to different choices of basis.
2. Let G be the group D_6 of syymetries of the equilateral triangle.



- Conjugates of id: just id itself.
- Conjugates of R_{120}: R_{120} and R_{240}.
  If T is a reflection, then
  $T \circ R_{120} \circ T^{-1} = R_{240}$.
- Conjugates of T_L: T_L, T_M and T_N.
  $R_{120} \circ T_L \circ R_{120}^{-1} = T_N$.
  $R_{240} \circ T_L \circ R_{240}^{-1} = T_M$.

Conjugacy is an equivalence relation

In a group G, write $x \sim y$ to mean that y is a conjugate of x, i.e. that $y = gxg^{-1}$ for some $g \in G$.

- $\sim$ is reflexive: $x \sim x$ for all $x \in G$, since $x = xxx^{-1}$.
- $\sim$ is symmetric
  Suppose $x \sim y$ and write $y = gxg^{-1}$, where $g \in G$.
  Then $x = g^{-1}yg = g^{-1}y(g^{-1})^{-1}$, so $y \sim x$ and $\sim$ is symmetric.
- $\sim$ is transitive.
  Suppose $x \sim y$ and $y \sim z$. Then we can write $y = gxg^{-1}$ and $z = hyh^{-1}$, where $g, h \in G$. We want to show $x \sim z$. Note

  $$z = hyh^{-1} = h(gxg^{-1})h^{-1} = hgxg^{-1}h^{-1} = (hg)x(hg)^{-1}.$$

  So $x \sim z$ and $\sim$ is transitive.

Challenge 1, Week 6

In the proof that conjugacy is an equivalence relation, we used the fact that the inverse of the product of two elements in a group is the product of their inverses in the opposite order, i.e.

$$(gh)^{-1} = h^{-1}g^{-1}.$$

This challenge has two parts.

1. Explain why this statment is true.
2. Think of an example from everyday life (not from mathematics) that shows that the inverse of composing two operations is to apply their inverses in the opposite order.