

Conjugacy Classes

Recall that elements x and y in a group G are **conjugate** to each other if

$$y = gxg^{-1} \text{ for some } g \in G.$$

The set of elements of G that are conjugate to x in G is called the **conjugacy class** of x , which we denote here by $\text{Cl}(x)$. Because conjugacy is an equivalence relation on G , conjugacy classes of different elements are either equal or disjoint:

- ▶ $\text{Cl}(x) = \text{Cl}(y)$ if x and y are conjugates of each other.
- ▶ $\text{Cl}(x) \cap \text{Cl}(y) = \emptyset$ if x and y are not conjugates of each other.

Note For $x \in G$, $\text{Cl}(x) = \{x\}$ if and only if $x \in Z(G)$.

An element of the centre of G has only one conjugate in G , namely itself.

How many conjugates does x have? (Theorem 2.2.9)

Let $x \in G$. Then every element of the form gxg^{-1} , where $g \in G$, is a conjugate of x in G , but they need not be distinct.

When is $g_1xg_1^{-1} = g_2xg_2^{-1}$?

$$\begin{aligned}g_1xg_1^{-1} &= g_2xg_2^{-1} \\ \iff g_1x &= g_2xg_2^{-1}g_1 \\ \iff g_2^{-1}g_1x &= xg_2^{-1}g_1 \\ \iff (g_2^{-1}g_1)x &= x(g_2^{-1}g_1)\end{aligned}$$

So the conjugates $g_1xg_1^{-1}$ and $g_2xg_2^{-1}$ are the same element if and only if $g_2^{-1}g_1 \in C_G(x)$. This occurs if and only if g_1 and g_2 belong to the same left coset of $C_G(x)$ in G .

Conclusion The number of distinct conjugates of x in G is $[G : C_G(x)]$, the index in G of the centralizer of x . Note that this means the number of elements in any conjugacy class is a divisor of $|G|$.

The Class Equation

Let G be a finite group. Then the number of elements of G is the sum of the numbers of elements of all of the conjugacy classes in G , all of which are divisors of $|G|$. This can be written as

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)],$$

where there is one x_i from each class with more than one element.

Consequence If the order of G is a prime power (like 3^7 or 5^3), the center of G cannot be trivial.

Suppose $|G| = p^k$ for some prime p . Then

$$p^k = |Z(G)| + \sum_i [G : C_G(x_i)].$$

We know that $|Z(G)| \geq 1$, we want to show $|Z(G)| \neq 1$. Every other term in the above equation is a multiple of p , so $Z(G)$ must be a multiple of p too, and $Z(G)$ is not the trivial subgroup of G .

Week 6, Challenge 2

In D_{2n} , the group of symmetries of the regular n -gon, how many conjugacy classes consist of reflections?

Give an explanation for your answer.

Note the answer depends on n , but in a way that can be described fairly easily.