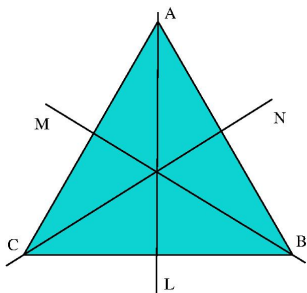


The Centre of Group

Definition Let G be a group with operation \star . The *centre* of G , denoted by $Z(G)$ is the subset of G consisting of all those elements that commute with every element of G , i.e.

$$Z(G) = \{x \in G : x \star g = g \star x \text{ for all } g \in G\}.$$

Example The centre of D_6 can be read from the group table. It consists only of the identity element.



\circ	id	R_{120}	R_{240}	T_L	T_M	T_N
id	id	R_{120}	R_{240}	T_L	T_M	T_N
R_{120}	R_{120}	R_{240}	id	T_M	T_N	T_L
R_{240}	R_{240}	id	R_{120}	T_N	T_L	T_M
T_L	T_L	T_N	T_M	id	R_{240}	R_{120}
T_M	T_M	T_L	T_N	R_{120}	id	R_{240}
T_N	T_N	T_M	T_L	R_{240}	R_{120}	id

Notes and Examples

- ▶ The centre of a group G is equal to G if and only if G is abelian.
- ▶ The centre of G is usually denoted by $Z(G)$.
- ▶ The identity element always belongs to the centre (this is built into the group axioms).
- ▶ The centre of D_{2n} consists only of the identity if n is odd, and also includes the rotation through 180° if n is even.
- ▶ The center of the general linear group $GL(2, \mathbb{R})$ is the set of all invertible **scalar matrices**, which have the same non-zero entry in both positions on the main diagonal, and zeros in the other positions.

Theorem 2.2.3 $Z(G)$ is a subgroup of G

Proof

1. $\text{id}_G \in Z(G)$, from the group axioms.

2. $Z(G)$ is closed under the operation \star of G .

Suppose $x, y \in Z(G)$ and let $g \in G$. We must show $(x \star y) \star g = g \star (x \star y)$. We know

$$\begin{aligned}(x \star y) \star g &= x \star (y \star g) = x \star (g \star y) \\ &= (x \star g) \star y = (g \star x) \star y = g \star (x \star y).\end{aligned}$$

3. Suppose $x \in Z(G)$. We must show $x^{-1} \in Z(G)$ also.

Let $g \in G$. Then

$$\begin{aligned}x \star g = g \star x &\implies x^{-1} \star (x \star g) \star x^{-1} = x^{-1} \star (g \star x) \star x^{-1} \\ &\implies g \star x^{-1} = x^{-1} \star g\end{aligned}$$

so $x^{-1} \in Z(G)$.