The centralizer of an element

Definition Let G be a group with operation $\star$, and let $x \in G$. The *centralizer* of x in G, denoted by $C_G(x)$, is the subset of G consisting of all those elements of G that commute with x under $\star$,

$$C_G(x) = \{y \in G : x \star y = y \star x\}.$$

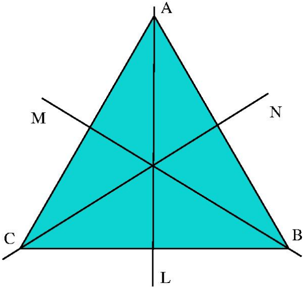
Notes

- $C_G(x)$ is a subgroup of G. (Exercise: prove this)
- It is easy to confuse the concepts of centre and centralizer. The centre belongs to the whole group. A centralizer belongs to a particular element.
- For every element x of G, $Z(G) \subseteq C_G(x)$.
  The centre of G is the intersection of the centralizers of all the elements of G.

Example

Let G be D_6, the symmetry group of the equilateral triangle. We can read the element centralizers from the group table.

- $C_G(\mathrm{id}) = \{\mathrm{id}, R_{120}, R_{240}, T_L, T_M, T_N\} = G$
- $C_G(R_{120}) = \{\mathrm{id}, R_{120}, R_{240}\}$ - the subgroup of rotations
- $C_G(R_{240}) = \{\mathrm{id}, R_{120}, R_{240}\}$ - the subgroup of rotations
- $C_G(T_L) = \{\mathrm{id}, T_L\}$, $C_G(T_M) = \{\mathrm{id}, T_M\}$, $C_G(T_N) = \{\mathrm{id}, T_N\}$.



$\circ$	id	R_{120}	R_{240}	T_L	T_M	T_N
id	id	R_{120}	R_{240}	T_L	T_M	T_N
R_{120}	R_{120}	R_{240}	id	T_M	T_N	T_L
R_{240}	R_{240}	id	R_{120}	T_N	T_L	T_M
T_L	T_L	T_N	T_M	id	R_{240}	R_{120}
T_M	T_M	T_L	T_N	R_{120}	id	R_{240}
T_N	T_N	T_M	T_L	R_{240}	R_{120}	id

A remark about towers of subgroups

Let G be a non-abelian group (it is only for non-abelian groups that centralizers are of interest).
Let x be an element of G that is not in the centre.
Then

- $C_G(x)$ is a proper subgroup of G.
- $Z(G)$ is a proper subgroup of $C_G(x)$.
- So we have a chain of proper inclusions

$$Z(G) \subsetneq C_G(x) \subsetneq G.$$

If the groups in question are finite, it follows that the index of $Z(G)$ in G cannot be prime.

Challenge for week 5

Let x be an element of a group G. Define

$$H(x) = \{y \in G : C_G(x) \subseteq C_G(y)\}.$$

Show that $H(x)$ is a subgroup of G.

Remark: The criterion for membership of $H(x)$ is that an element belongs to $H(x)$ if everything that commutes with x also commutes with that element. You can solve this challenge using the usual logical framework for showing that something is a subgroup, or you can do it by showing that $H(x) = Z(C_G(x))$.