Lecture 7: Lagrange's Theorem

Theorem

(Lagrange's Theorem) Let G be a finite group with a subgroup H. Then the order of H divides the order of G.

Notes

1. Recall that "divides" means "is a factor of". The symbol for "divides" is a vertical bar. For example "$3|21$" is the statement that 3 is a divisor of 21.
2. Lagrange's Theorem says that a subgroup of S_4, which has $4! = 24$ elements, could possibly have $1, 2, 3, 4, 6, 8, 12$ or 24 elements, but couldn't have (for example) 7 or 16 elements.
3. The converse of Lagrange's Theorem is not true; if n and k are integers and $k|n$, it is not true that every group of order n has a subgroup of order k.

Left cosets

Definition Let H be a subgroup of a group G (with binary operation $\star$). Then the left coset of H in G determined by x, which is denoted xH or $x \star H$, is the set $xH = \{x \star h : h \in H\}$.

Notes

1. xH consists of the elements of H, all “translated” by being composed on the left with the element x. We can think of it as a “shifted copy” of H inside G.
2. xH is a subset of G, generally not a subgroup.
3. It is possible for two different elements x and y of G to determine the same left coset of H. For example if x is any element of the subgroup H, then xH is just H itself.
4. There is a corresponding concept of right coset, which we will care about later but not now. The right coset of H determined by x would be $Hx = \{hx : h \in H\}$.

Examples of Left Cosets

If G is the group of integers under addition, and H is a subgroup $5\mathbb{Z}$ consisting of all multiples of 5, then the left coset of H in G determined by 3 is

$$\begin{aligned} 3 + H &= \{\dots, 3 + (-5), 3 + 0, 3 + 5, 3 + 10, \dots\} \\ &= \{\dots, -2, 3, 8, 13, \dots\} \\ &= 3 + 5\mathbb{Z}. \end{aligned}$$

This is the congruence class of 3 modulo 5. Note that $3, 8, -2, -7$ all determine the same coset (they are all congruent to each other modulo 5).

Examples of Left Cosets

If G is $\mathrm{GL}(2, \mathbb{Q})$, the group of all 2×2 rational matrices under matrix multiplication, and H is the subgroup $\mathrm{SL}(2, \mathbb{Q})$ consisting of all matrices of determinant 1, then the left coset of H in G determined by $\begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}$ is

$$\left\{ \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix} B : \det(B) = 1 \right\}.$$

This set consists of all matrices in $\mathrm{GL}(2, \mathbb{Q})$ whose determinant is -6.

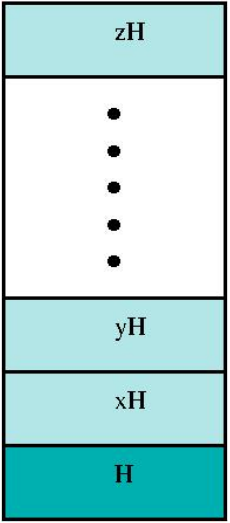
Examples of Left Cosets

If G is the dihedral group D_8 consisting of symmetries of the square, and H is the subgroup of G consisting of the four rotations, then the left coset of H in G determined by any one of the four reflections consists of the four reflections. The left coset of H in G determined by any one of the four rotations consists of the four rotations.

Note that there are only two distinct cosets, they have empty intersection and their union is the whole group G.

Proof Mechanism for Lagrange's Theorem

Start with the subgroup H of the finite group G.
If $H = G$ the theorem holds.
If not, choose an element x of G with $x \notin H$.
Then the coset xH is disjoint from H and has $|H|$ elements.
If $H \cup xH = G$ then $|G| = 2|H|$ and we are done.
If not, choose $y \notin H \cup xH$ and add the coset yH.
Eventually we find that G is the union of k disjoint left cosets of H, and $|G| = k|H|$.



Challenge for Week 4

The Euclidean plane $\mathbb{R}^2$ is a group under vector addition. The elements are ordered pairs (a, b) (points in the plane) and the addition is defined by $(a, b) + (c, d) = (a + c, b + d)$. The group $\mathbb{R}^2$ is also a real vector space, which means that its elements can be multiplied by real numbers as well as added together.

What are the nontrivial proper subgroups of $\mathbb{R}^2$ that are also closed under multiplication by real numbers?
If H is such a subgroup of $\mathbb{R}^2$, what do the left cosets of H determined by different elements of $\mathbb{R}^2$ look like?

Hint/Remark: There are very many nontrivial proper subgroups of $\mathbb{R}^2$ of this type, but geometrically they all look alike, and their left cosets have a nice geometric description too. A good answer to this challenge would be a geometric explanation with a picture.