MA3343: Workshop Week 2

Welcome

October 9, 2020

Problem (Autumn 2020)

Determine whether the set of positive real numbers is a group under the operation $\star$ defined by $x \star y = x^y$.

1 Is $*$ is a well-defined binary operation on $\mathbb{R}_{>0}$? For every pair of elements x, y of $\mathbb{R}_{>0}$, does x^y exist and is it in $\mathbb{R}_{>0}$?

Yes ✓

2. Is $*$ associative? For all $x, y, z \in \mathbb{R}_{>0}$,

$$\boxed{(x * y) * z \stackrel{?}{=} x * (y * z)}$$

$$(x^y)^z \stackrel{?}{=} x^{(y^z)}$$

$$\overset{\shortparallel}{x^{yz}} \neq x^{y^z} \text{ in general}$$

$$\boxed{\begin{array}{l} * \text{ is not associative} \\ \Rightarrow \text{NOT a group} \end{array}}$$

e.g. $(2^3)^4 = 2^{12} \neq 2^{(3^4)} = 2^{81}$

Alternatively

2(e) Does $\mathbb{R}_{>0}$ have an identity element for $*$?

Such an element b would have to satisfy

$\boxed{b * x} = x$ and $\boxed{x * b} = x$ for all x

$b^x = x$ and $\boxed{x^b = x}$ for all x

no value of b works here $\Longleftarrow$ $b = 1$?

No identity element $\Longrightarrow$ Not a group.

Groups of Permutations

A permutation of a finite set S is a function from S to S that sends every element to a different image.

For example

$$a \to b, \ b \to d, \ c \to c, \ d \to a$$

bdca

is a permutation of the set $\boxed{\{a, b, c, d\}}$.

You can think of a permutation as a rearrangement of the elements, but if we think of them as functions, they come equipped with a binary operation, namely composition of functions.

Under this operation, the set of all $n!$ permutations of a set of n elements is a group, called the symmetric groups of degree n and denoted S_n. It is conventional to let the set of n elements be $\{1, 2, \ldots, n\}$ unless there is a reason to do otherwise.

Composition of Permutations in S_4

$$\sigma: \begin{array}{l} 1 \to 3 \\ 2 \to 1 \\ 3 \to 4 \\ 4 \to 2 \end{array} \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

sigma

$$\tau: \begin{array}{l} 1 \to 3 \\ 2 \to 2 \\ 3 \to 4 \\ 4 \to 1 \end{array}$$

Tau

$\sigma \circ \tau$ – σ "after" τ

$$\begin{array}{ccccc} 1 & \to & 3 & \to & 4 \\ 2 & \to & 2 & \to & 1 \\ 3 & \to & 4 & \to & 2 \\ 4 & \to & 1 & \to & 3 \\ & \tau & & \sigma & \end{array}$$

$\sigma \circ \tau$

$$\sigma \circ \tau : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$\tau \circ \sigma \neq \sigma \circ \tau$

not commutative.

$$\tau \circ \sigma : \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}}_{\tau} \circ \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}}_{\sigma} = \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}}_{\tau \circ \sigma}$$

- Composition is associative

Function composition is always associative

$$((f\circ g)\circ h)(x) = (f\circ(g\circ h))(x) \quad \text{always}$$

$$(f\circ g)(h(x)) \overset{?}{=} f((g\circ h)(x))$$

$$f(g(h(x))) = f(g(h(x))) \checkmark$$

- Identity element $\boxed{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}}$ Identity permutation

- Inverses

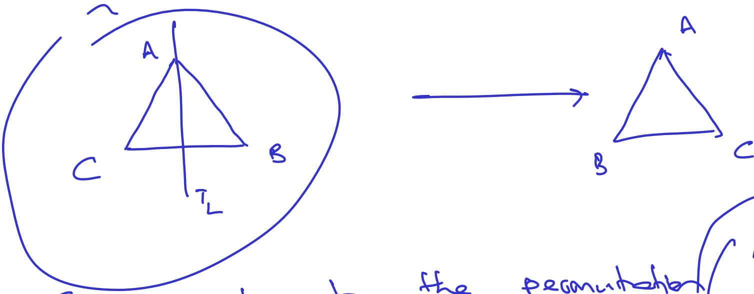
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

σ : $1 \rightarrow 3$, $2 \rightarrow 2$, $3 \rightarrow 4$, $4 \rightarrow 1$

σ^{-1} : $3 \rightarrow 1$, $2 \rightarrow 2$, $4 \rightarrow 3$, $1 \rightarrow 4$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

Connection to symmetries of $\triangle$



Corresponds to the permutation $\begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}$ of the vertices

Weekly challenges

Thank you everyone for all the responses!

- Deadline (for assessment) is 5pm on Friday of the following week.
- On the Blackboard gradebook, every submission will get a score out of 10 (translated into 2% of overall mark).
  - 10 for a complete answer with a high quality of presentation, that would communicate the idea really well to a reader in the class;
  - 7 for a well-presented almost complete answer or for a complete answer that is not perfectly communicated
  - 4 for an answer that has some substantial relevant info but is missing something substantial.
  - Less than that if the point is missed or misinterpreted.
- Five excellent answers to each challenge will be posted on the course website (anonymised by default). Authors will vary from week to week.