Lecture 3: Groups of permutations and symmetries

A permutation of a finite set T is a *bijection* from T to T. This means that every element has a different image, and the image of the function is the whole set T. The permutations of a set form a group under composition.

The group of all permutations of a set of n elements is called the symmetric group of degree n and denoted S_n.

How the composition operation works: the example of S_3.

S_3: the set of all permutations of $\{1,2,3\}$

Take σ, τ in S_3

(σ: sigma, τ: tau)

σ: $1 \to 3$, $2 \to 1$, $3 \to 2$

τ: $1 \to 2$, $2 \to 1$, $3 \to 3$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

(Notation)

Composition

$\sigma \circ \tau$: σ "after" τ

$$1 \xrightarrow{\tau} 2 \xrightarrow{\sigma} 1$$
$$2 \xrightarrow{\tau} 1 \xrightarrow{\sigma} 3$$
$$3 \xrightarrow{\tau} 3 \xrightarrow{\sigma} 2$$

$\sigma \circ \tau$: $1 \to 1$, $2 \to 3$, $3 \to 2$

$$\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

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$\tau \circ \sigma$

τ after σ

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}}_{\tau} \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}}_{\sigma} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Note $\tau \circ \sigma \neq \sigma \circ \tau$

Identity permutation: $id = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$\sigma \circ id = \sigma$

$id \circ \sigma = \sigma$

for any $\sigma \in S_3$

Inverses Two permutations σ, τ are inverses of each other if $\sigma \circ \tau$ and $\tau \circ \sigma$ are both equal to id.

The inverse of σ:

$$\sigma: \quad \begin{array}{ccc} & \sigma & \\ 1 & \rightleftarrows & 3 \\ 2 & \rightleftarrows & 2 \\ 3 & \rightleftarrows & 1 \\ & \sigma^{-1} & \end{array}$$

In "array notation"

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad \sigma^{-1} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

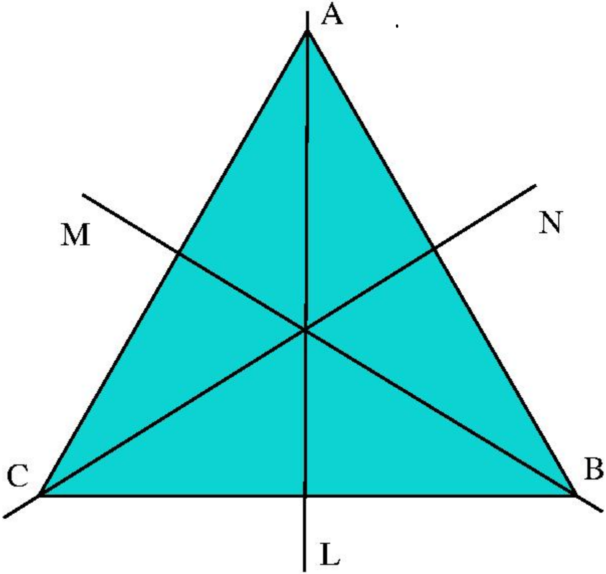
Example

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

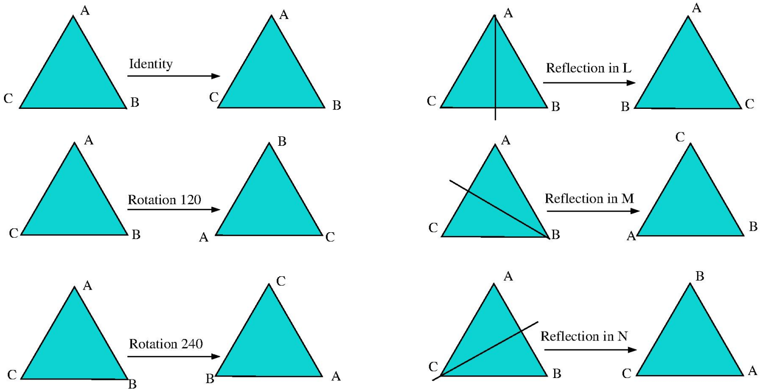
$$\tau^{-1} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Note $\tau^{-1} \neq \tau$ in this case.

The group of symmetries of the equilateral triangle



Symmetries of the triangle

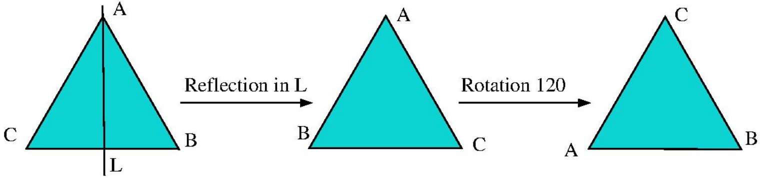


The Group Operation in D_6

The group D_6 of symmetries of the triangle has six elements.

$$D_6 = \{\mathrm{id}, R_{120}, R_{240}, T_L, T_M, T_N\}.$$

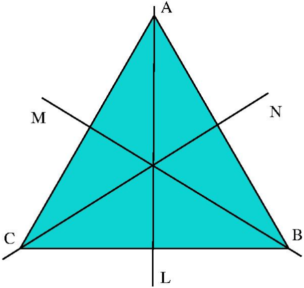
The group operation is composition, denoted by the symbol $\circ$. $R_{120} \circ T_L$ means "R_{120} *after* T_L, the symmetry obtained by applying T_L first and then R_{120}. We can figure out which one it is by watching what happens to the vertices in this composition of symmetries.



Comparing the final position to the starting position, we see that

$$R_{120} \circ T_L = T_M.$$

Group table for D_6



$\circ$	id	R_{120}	R_{240}	T_L	T_M	T_N
id	id	R_{120}	R_{240}	T_L	T_M	T_N
R_{120}	R_{120}	R_{240}	id	T_M	T_N	T_L
R_{240}	R_{240}	id	R_{120}	T_N	T_L	T_M
T_L	T_L	T_N	T_M	id	R_{240}	R_{120}
T_M	T_M	T_L	T_N	R_{120}	id	R_{240}
T_N	T_N	T_M	T_L	R_{240}	R_{120}	id

In general, the group of symmetries of the regular n-gon is denoted D_{2n} and called the dihedral group of order $2n$. It has $2n$ elements, n rotations (including the identity) and n reflections.

Weekly Challenge 2

Like a polygon, a 3-dimensional object has a group of symmetries, which includes rotations and reflections. This week's challenge is to give a description of the rotational symmtries of the cube. How many are there? What are the axes about which rotational symmetries occur, and what the the angles of rotation?