Quantum Mechanics: The Stabilizer Formalism

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Introduction

The stabilizer formalism of quantum mechanics presents a novel way of describing the machinery of quantum mechanics using concepts from Group Theory. The stabilizer formalism uses the concepts of the stabilizer of a group and generators of a group in order to characterise quantum states and the result of operations on those states.

Quantum States

- Quantum states are represented as unit vectors in a complex vector space. The state is represented using a set of orthonormal basis vectors which span the space. Let $|0\rangle$ and $|1\rangle$ represent a basis of two orthonormal vectors in $\mathbb{C}^2$ such that:

  $$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

  Thus, a quantum state, $|\psi\rangle \in \mathbb{C}^2$, could have the form:

  $$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$
- A quantum state in $\mathbb{C}^2$ is called a qubit state. The basis of quantum computing consists in manipulating qubit states through matrix-vector multiplication.

Multiple Qubit States

- The tensor product operation combines multiple qubits into a single composite system. The composite system of two qubits, $|\psi\rangle \otimes |\psi\rangle$, occupies the complex vector space $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$. Thus, a 2-qubit state can be represented using a set of four orthonormal basis vectors spanning the space $\mathbb{C}^4$.
- In general, a composite system of n qubits is defined in $\mathbb{C}^2_1 \otimes \mathbb{C}^2_2 \otimes ... \otimes \mathbb{C}^2_n = \mathbb{C}^{2^n}$. An arbitrary n-qubit state can be represented using an orthonormal basis consisting of 2^n vectors which span the vector space $\mathbb{C}^{2^n}$.

The Pauli Group

- Operators are matrices which act on quantum states. Operators acting on separate qubit states are also combined using the tensor product operation. For two operators U_1 acting on a qubit $|\psi_1\rangle$ and U_2 acting on another qubit $|\psi_2\rangle$, the total action on the composite system is simply:

  $$(U_1 \otimes U_2)(|\psi_1\rangle \otimes |\psi_2\rangle) = (U_1 |\psi_1\rangle \otimes U_2 |\psi_2\rangle)$$
- For the qubit state, $|\psi\rangle \in \mathbb{C}^2$, there exist a special class of operators known as the Pauli matrices:

  $$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
- A group under the operation of matrix multiplication, known as the Pauli group, G_1, can be defined using these operators with the multiplicative factors ± 1 and $\pm i$:

  $$G_1 = \{\pm\mathbb{I}, \pm i\mathbb{I}, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$$

  The Pauli group generalizes to the n-qubit case, where each element of the Pauli group, G_n, is a distinct tensor product of n individual Pauli matrices with multiplicative factors ± 1 and $\pm i$[1].
- Every matrix in the group G_1 can be *generated* by the elements X,Y and Z:

  $$G_1 = < X, Y, Z >$$

The Stabilizer Formalism

- Consider a subgroup S of G_n, and define the subspace V_s to be the set of n-qubit states which are fixed by every element of S. The set of vectors V_s is *stabilized* by S and the subgroup S forms the *stabilizer* of the subspace V_s, $Stab_{G_n}(V_s)$.
- The subgroup S can be defined in terms of its *generators*. To assess whether a particular n-qubit state belongs to the stabilized subspace, V_s, it suffices to check whether the vector is stabilized by the generators of S[1].

Arbitrary Operations

- Quantum operators are unitary, this means they satisfy the relation $UU^\dagger = U^\dagger U = \mathbb{I}$ where U is a unitary operator and $U^\dagger$ involves transposing U and complex conjugating all of the entries.
- Consider an arbitrary unitary operator, U, applied to the vector space V_s which is stabilized by the subgroup S. For any vector $|\psi\rangle \in V_s$ and matrix element $g \in S$ we find:

  $$U|\psi\rangle = Ug|\psi\rangle = (UgU^\dagger)U|\psi\rangle$$
- The new vector $U|\psi\rangle$ is stabilized by $UgU^\dagger$. After applying the operator U to our entire vector space V_s, we can see that our new vector space UV_s has stabilizer $USU^\dagger = \{UgU^\dagger | g \in S\}$.
- If the stabilizer S has generators $< g_1, g_2, ..., g_n >$ then our new stabilizer, $USU^\dagger$, has generators $< Ug_1U^\dagger, Ug_2U^\dagger, ..., Ug_nU^\dagger >$.

Benefits of the Stabilizer Approach

- It can be shown that the Pauli Z gate stabilizes the $|0\rangle$ state i.e. $Z|0\rangle = |0\rangle$. Thus, an n-qubit state:

  $$|\phi\rangle = (|0\rangle_1 \otimes |0\rangle_2 \otimes ... \otimes |0\rangle_n) = |0\rangle^{\otimes n} \in \mathbb{C}^n$$

  has a stabilizer with a single element, $S = \{Z_1 \otimes Z_2 \otimes ... \otimes Z_n\}$.
- Define a unitary operator, H, with the property $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Applying this operator to $|\phi\rangle$ gives:

  $$H^{\otimes n}|\phi\rangle = H_1|0\rangle_1 \otimes ... \otimes H_n|0\rangle_n \in \mathbb{C}^{2^n}.$$
- However, the Pauli X gate stabilizes this resulting state:

  $$X\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
- This means our new state $H^{\otimes n}|\phi\rangle$ also has a single-element stabilizer, $S' = \{X_1 \otimes X_2 \otimes ... \otimes X_n\}$. Our stabilizer still has n terms, but the vector representation of our new state has 2^n terms! For 100 qubits, this is a $\approx 10^{28}$ improvement...
- A group-theoretic approach to simple quantum computations allows for classical simulations as we can keep the number of terms *linear* in n.

References

Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition.* Cambridge University Press, 2010. DOI: `10.1017/CBO9780511976667`.