Generating Sets of Finite Symmetric Groups

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Introduction

This poster serves as an after-class reading for MA3343 students. We assume the readers are familiar with the concepts of groups, generating sets, symmetric groups, cycles and transpositions. In this poster, we investigate the topic of generating sets of finite symmetric groups. In particular, we focus on the number of elements in the generating sets and find the smallest such number by mathematical reasoning.

1. S_n is generated by its all cycles

Every permutation can be written as a product of cycles, **so the set of all cycles in S_n generates S_n**. However, such set is very large and redundant. Take S_6 for example, we have $\{(1), ..., (6), (12), (13), , , , (16), (123), ...\}$, a generating set with 415 elements, and clearly not all cycles are distinct.

2. S_n is generated by its transpositions

Recall that every cycles can be written as a product of transpositions. For instance, (123) = (12)(23). Since the set of all cycles generates S_n, **the set of all transpositions is a generating set of S_n**. The total number of transpositions in S_n is $\binom{n}{2} = \frac{n(n-1)}{2}$[1]. It is a much smaller set compared the set of all cycles, but it is still redundant as (23) can be written as (12)(13)(12).

3. S_n is generated by $n-1$ transpositions

- **S_n is generated by (12)(13)(14)...(1n)**.
  Since S_n is generated by the set of all its transpositions, it is sufficient to show that an arbitrary transposition (ij) can be written as a product of transpositions containing 1[1]. Consider (1i)(1j)(1i). Element i is sent to 1 and then to j, and element j is sent to 1 and then to i. Thus, (ij) = (1i)(1j)(1i) for all $1 < i \neq j \leq n$, as required.
- **S_n is generated by (12)(23)(34)..(n-1 n)**.
  Following the above result, we only need to show that for all $1 < i \leq n$, (1i) is a product of adjacent transpositions.
  We use mathematical induction.
  •**base case** Clearly our claim holds for (12).
  •**induction step** Suppose (1i) can be written as a product of transpositions swapping 1 with other elements.
  Then (1 i+1) = (1i)(i i+1)(1i), and the result follows.
- **In both cases, we have $n-1$ elements in generating sets**.

Before moving on...

We have already shown that S = $\{(12), (13), ..., (1n)\}$ is a generating set for S_n. We may ask ourselves..

Q1. **Is S non-redundant?**
A1. **YES**. If we remove any element from S, it is no longer a generating set. This is not hard to see, as removing (1i) from S leaves us a fixed point i because none of the remaining transpositions permute i. A generating set has such property is called '*Minimal*'.

Q2.**Is S the smallest generating set for S_n?**
A2. **NO**. A minimal generating set is non-redundant, but is not necessarily of the minimum size. We will see in the next section that, though S is minimal, the minimum size of generating sets for S_n where $n \geq 3$ is 2.

4. The minimum cardinality of generating sets for S_n

- S_2 is generated by (12), as (1)(2) = (12)(12) and (12) = (12).
- **For $n \geq 3$, S_n is generated by the transposition (12) and the n-cycle (123...n)** [1].
  Since S_n is generated by (12)(23)...(n-1 n), it suffices to show that (i-1 i) where $2 \leq i \leq n$ can be generated by (12) and (123...n).
  We use mathematical induction.
  •**base case** Clearly (12) is generated by (12) and (123...n).
  •**induction step** Suppose (i-2 i-1) is generated by (12) and (123..n). We want to show (i-1 i) is also generated by these two cycles.
  For convenience, we denote (123..n) by σ.
  Consider $\sigma(i-2 \;\; i-1)\sigma^{-1}$, an element generated as requirement.
  Then, $\sigma(i-2 \;\; i-1)\sigma^{-1} = (\sigma(i-2) \;\; \sigma(i-1)) = (i-1 \;\; i)$, and the result follows.
- **For $n \geq 3$, S_n cannot be generated by single element**.
  Recall the group generated by single elements is cyclic. Since every cyclic group is abelian, it suffices to show S_n is non-abelian.
  Let permutations $\pi_1 = (13) \in S_n$ and $\pi_2 = (12) \in S_n$.
  Then $\pi_1\pi_2 = (231)$ and $\pi_2\pi_1 = (132)$.
  It follows that S_n in non-abelian as $\pi_1\pi_2 \neq \pi_2\pi_1$.
- **Combining the above results, we conclude that the minimum cardinality of generating sets for S_n where $n \geq 3$ is 2**.

References

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