Rotational Symmetries Of A Cube

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Overview

We will look at the following aspects of the Group of Rotational Symmetries of a Cube.

- Showing that the rotations of a cube form a group.
- Illustrating the various rotations and axes of the rotations of a cube.
- Proving the group obeys the orbit stabilizer theorem.
- Showing the group of rotations of a cube is isomorphic to S_4

Prove that the Rotations of a Cube form a Group

Here G is the set of rotations of a cube and let x be some arbitrary rotation of the cube, i.e x in G.

- Identity
  We need to show that there is an element id such that id * x = x * id = x for x in G. we need any rotation which leaves the cube completely unchanged. The rotation by 360 degrees through any axis does this, so id in G
- Closure
  Let a, b in G. Must show a * b in G. Apply a to an unmarked cube simply rotates the cube about a certain axis, leaving a seemingly identical cube. Applying b then after this leaves an identical cube as well. So a * b in G
- Inverse
  Each rotation is the inverse of itself. If you apply rotation x to a cube in one direction and then rotate in the opposite direction by the same degree, you are left with the cube you started with. This is true for any rotation x in G. Thus inverse of x is in G.

Isomorphism to S_4

The group of rotations of a cube has the same order as S_4, 24. We need to show the group is isomorphic to a subgroup of S_4. Remark that a cube has four diagonals and that the rotation group induces a group of permutations on the four diagonals. However we must not assume that different rotations correspond to different permutations.
Now labelling the consecutive diagonals 1,2,3 and 4, we see two perpendicular axes where 90 degree rotations give the permutations $alpha = (1234)$ and $alpha = (1432)$. These induce an 8 element subgroup and a 3 element subgroup $\{id, \alpha, \alpha^2, \alpha^3, \beta^2, \beta^2\alpha, \beta^2\alpha^2, \beta^2\alpha^3\}, \{id, \alpha\beta, (\alpha\beta)^2\}$ respectively. Therefore, the rotations induce all 24 permutations since $24 = lcm(8, 3)$ [2].

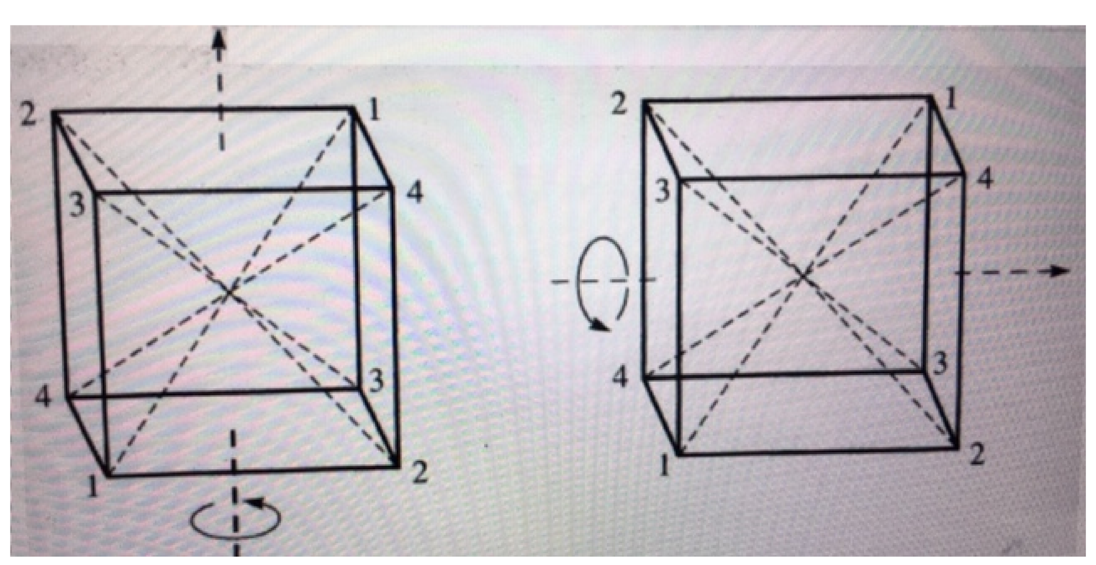

Fig. 1: $\alpha = (1, 2, 3, 4)\beta = (1, 4, 3, 2)$

Rotations and Axes of Rotation

The cube has a total of 24 rotations. The first rotation is the identity Now lets look at the diagram below on the left. We have nine rotations, we can rotate by $90°, 180°$, or $270°$, around each of the three axes shown. Each of these nine rotations will leave two faces fixed, and all vertices and edges are not fixed.The middle diagram shows six more rotations of the cube. These six leave two edges fixed, while no faces or vertices are fixed. The final diagram below shows rotations around the axes made by joining opposite vertices, we can rotate by $120°, 240°$ about these four axes, resulting in eight more rotations.

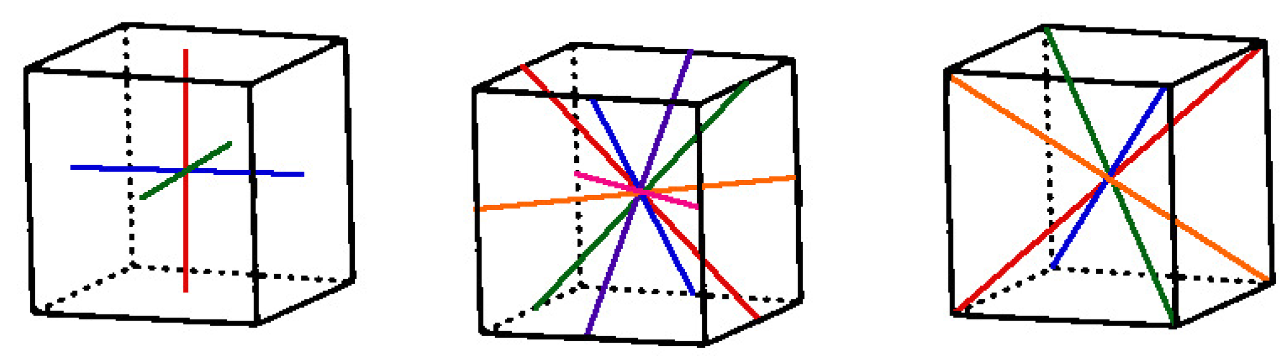

Fig. 2: Rotational Axis of a cube

[1]

The Orbit-Stabilizer Theorem

We let f = a face in a cube
|G . f] = |G : Stab$_G$(f)|
(Aside: We note that there are 24 rotational symmetries in a cube i.e. |G| = 24)

Proof:
If we take any face on a cube, it is possible to move from that face to any other face in the cube (as seen in the first figure above).We then have that the orbit of any face is $\{1, 2, 3, 4, 5, 6\}$ (the numbers 1 to 6 represent a face on the cube).The stabilizer of all faces in a cube are $\{id, R_{90}, R_{180}, R_{270}\}$ (as seen in the first image (left) above and we then apply the appropriate axis in each case).
|G . f| = 6, |G : Stab$_G$(f)| = |24 : 4| = 6
6 = 6

We now take a look at the edges.
We let e = an edge in a cube
|G . e] = |G : Stab$_G$(e)|
(Aside: Again we note that there are 24 rotational symmetries in a cube i.e. |G| = 24)

Proof:
If we take any edge on a cube, it is possible to move from that edge to any other edge in the cube. So, we then have that the orbit of any edge is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ (the numbers 1 to 12 represent an edge on the cube). The stabilizer of any edge on the cube is $\{id, R_{180}\}$ (where the axis of rotation goes from the centre of that edge to the centre of the edge on the face directly opposite to it, passing through the centre of the cube, as seen in the middle image above.
|G . e| = 12, |G : Stab$_G$(e)| = |24 : 2| = 12
12 = 12

Subgroups

We now take a look at the subgroups of the rotational symmetries of a cube.

- The stabilizer of each face forms a subgroup of order 4 and the stabilizer of each edge forms a subgroup of order 2.
- If we examine the first image (left) as seen under Rotations and Axis of Rotation, which shows the face midpoint rotations.For the axes shown, all rotations through the same axis form a subgroup of order 4.
- We now examine the middle image as seen under Rotations and Axis of Rotation, which shows the edge midpoint rotations. For the axes shown, all rotations through the same axis form a subgroup of order 2.
- Lastly, we examine the last image (right) as seen under Rotations and Axis of Rotation, which shows the diagonal rotations. For the axes shown, all rotations through the same axis form a subgroup of order 3.

The order of the subgroups are 2,3 and 4, which are factors of |G| = 24. This verifies Lagrange's theorem.

Is the Group Abelian?

Centre of the group, $Z(G) = \{id\}$, the centralizer of any element x of the group $C_G(x) =$ {id, any rotation on the same axis as that rotation x}. Hence the group is not abelian as not every element of the group commutes with one another.

References

[1] unknown. *The Rotational Symmetries of the Cube.* URL: `https://garsia.math.yorku.ca/~zabrocki/math4160w03/cubesyms/`. (accessed: 15.12.2020).

[2] unknown. *Theorem (The rotation Group of the Cube). The group of rotations of a cube is isomorphic to S4.* URL: `http://facstaff.cbu.edu/~wschrein/media/M402%20Notes/M402L104.pdf`. (accessed: 15.12.2020).