The Group of Symmetries of the Regular Tetrahedron

Evan O'Riordan   Laura O'Donnell

Introduction to the Group of Symmetries of the Regular Tetrahedron.

This is a study of the regular tetrahedron, a three-dimensional object that consists of four vertices A, B, C, D (we could also say 1, 2, 3, 4), six edges (AB), (AC), (AD), (BC), (BD), (CD), and four faces. Four vertices give us 4! = 24 permutations or **symmetries** in this group. There are 12 rotations and 12 reflections. What are these symmetries, and what do they look like?
Note that we will be representing our symmetries using array representation, where an element on the top row is being mapped to a corresponding element on the bottom row (see examples below).

The Identity Element

In group theory, the identity element is the element of the group that when combined with any other element under the group's binary operation, regardless of the order performed, returns that other element of the group.
For the symmetries of a tetrahedron, the identity element is the transformation that leaves all the vertices in the same place. It can be thought of as not moving the tetrahedron, or else as rotating the tetrahedron 360° about any of its axes.
With array representation (vertices {A,B,C,D}), the identity element is represented as

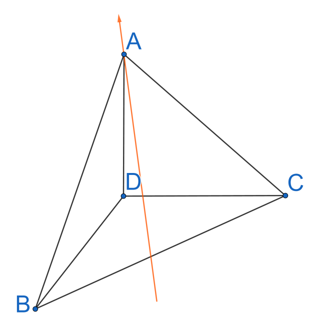
$$\begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix}$$

Rotation About the Axes at the Vertices

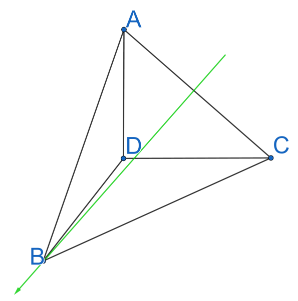
The figures below represent the four rotational axes of symmetry that connect a vertex to the center of its opposite face. For example, the axis at vertex A is the line that connects the vertex A to the center of the triangular face BCD.
Eight rotational symmetries can be found by rotating the tetrahedron by either 120° or 240° about each of these four axes. These rotations keep one vertex fixed and cycle the other three. They are as following:

	About A	**About B**	**About C**	**About D**
120°	$\begin{pmatrix} A & B & C & D \\ A & C & D & B \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ D & B & A & C \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ B & D & C & A \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ C & A & B & D \end{pmatrix}$
240°	$\begin{pmatrix} A & B & C & D \\ A & D & B & C \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ C & B & D & A \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ D & A & C & B \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ B & C & A & D \end{pmatrix}$

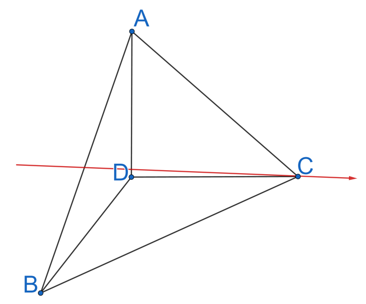
Axis at Vertex A



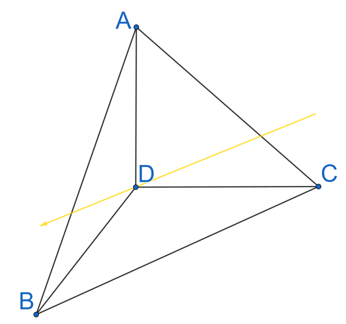
Axis at Vertex B



Axis at Vertex C



Axis at Vertex D



Rotation About the Edge Axes

Three more symmetries can be found by rotating the tetrahedron 180° along each of the axes that connect opposite edges through their midpoints, i.e. the lines connecting the midpoint of AB to CD, AC to BD, or AD to BC.
These rotations can be seen as swapping any two pairs of points. They are as follows:

AB to CD	$\begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix}$
AC to BD	$\begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix}$
AD to BC	$\begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$

These rotations can also be found by composing two of the vertex rotations with opposite rotations on different vertices (e.g. 120° about A followed by 240° about B).

Reflections of Two Vertices

We can find 6 of the tetrahedrons 12 reflections by taking a pair of vertices and swapping them, while leaving the other two fixed in place. These 6 reflections are as follows:

AB	**AC**	**AD**
$\begin{pmatrix} A & B & C & D \\ B & A & C & D \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix}$
BC	**BD**	**CD**
$\begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ A & B & D & C \end{pmatrix}$

Reflections composing of a Rotation & Reflection

The last 6 reflections can be found by combining one of the reflections with six of the rotations. We need to pick our rotations and reflections so that after composing the rotation and reflection none of the vertices are in their original position. Here are the six reflections and examples of how to find them:

(120° A) ∘ AB	**(120° B) ∘ AB**	**(240° A) ∘ AB**	**(240° B) ∘ AB**	**(AC to BD) ∘ AB**	**(AD to BC) ∘ AB**
$\begin{pmatrix} A & B & C & D \\ C & A & D & B \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ B & D & A & C \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ D & C & A & B \end{pmatrix}$	$\begin{pmatrix} A & B & C & D \\ C & D & B & A \end{pmatrix}$

The resulting symmetries are derangements of the four vertices. Notice how we used the reflection of AB alongside the four rotations about the vertex that kept either A or B fixed, as well as the two edge rotations that did not swap A with B. If we used any of the other rotations, at least one vertex would have stayed fixed, meaning we would not have uncovered a new reflection!

Additional Information & References

Armstrong, M. A. (1988). 'Symmetries of the Tetrahedron', *Groups and Symmetry.* New York: Springer, pp. 1-5.
Johnson, P. (2009), *Full Tetrahedral Symmetry.* viewed 6 December 2020, <http://patrickjohnson.name/311PROJ/Tetrahedron2.html>.
The figures used to represent rotations were created using Geogebra (www.geogebra.org).