The Group Of Symmetries Of The Tetrahedron

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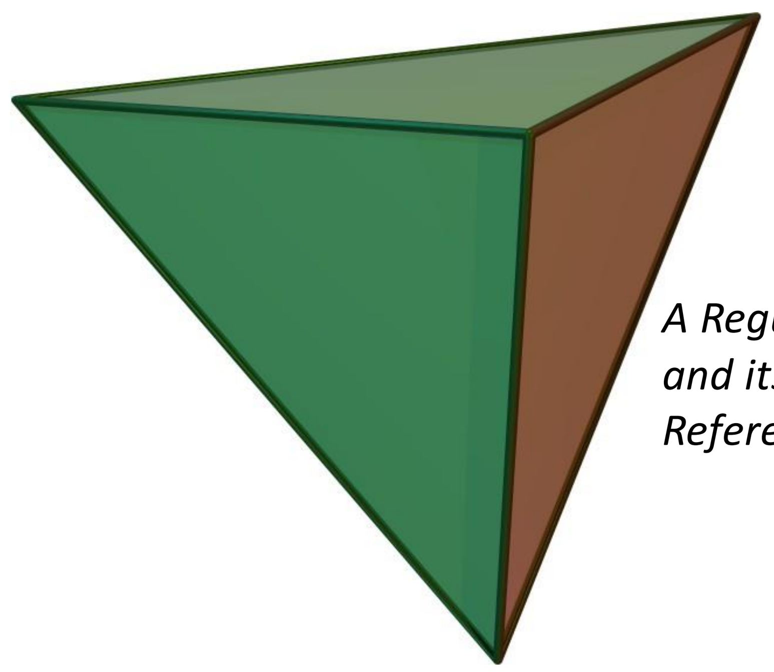
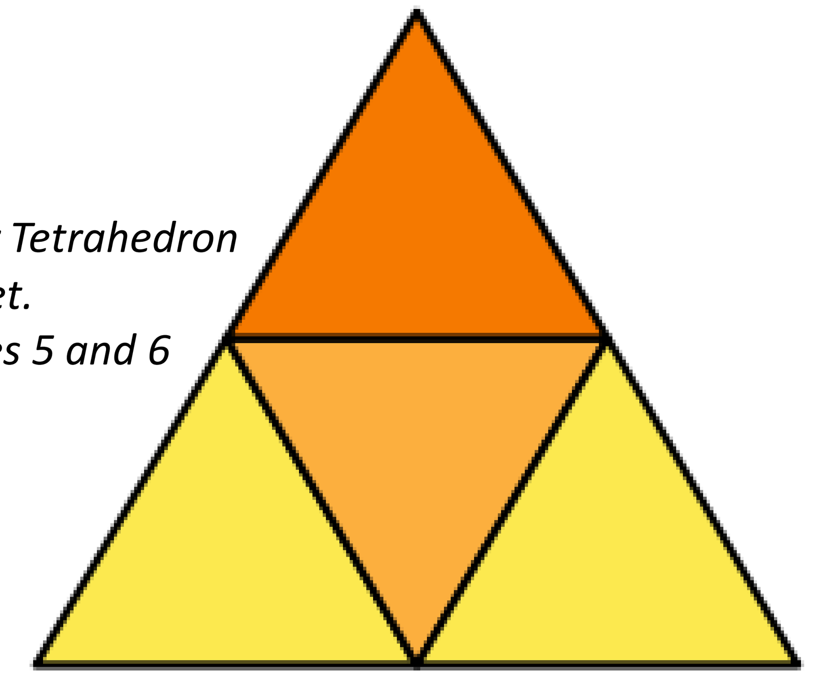
Introduction

This poster has been made by Brendan O'Donoghue, Cian Griffin, Paulina Karwan and Stephen Malone for our Groups module MA3343. We are all Mathematics and Education students, and we wanted to use this poster to educate everyone on the symmetries of a Regular Tetrahedron.





A Regular Tetrahedron and its Net. References 5 and 6

Properties and Facts of a Regular Tetrahedron

A Regular Tetrahedron (Also known as a Triangular Pyramid) is a pyramid which has four faces which are all equilateral triangles. Any of the four faces can be considered the base, this will be shown when we describe the symmetries of a Regular Tetrahedron.
A Regular Tetrahedron is a convex polyhedron which can be geometrically described as a polyhedron where any line connecting any two points on the surface of the shape all are contained within the interior of the polyhedron.
It is also one of the five regular Platonic solids which are derived from Eulers formula F(Faces) + V(Vertices) - E(Edges) = 2
A Regular Tetrahedron has Four Triangular Faces, Four Vertex Corners and Six Edges.
Tetrahedrons are represented in real life through chemistry with the structure of molecules (e.g., Methane) and in a four-sided dice which is used less commonly then the six-sided dice.
The group of symmetries of the tetrahedron has 24 elements and it is isomorphic to the symmetric group of degree 4 (the group of all permutations of four objects).

Regular tetrahedron
Solve for volume ▾

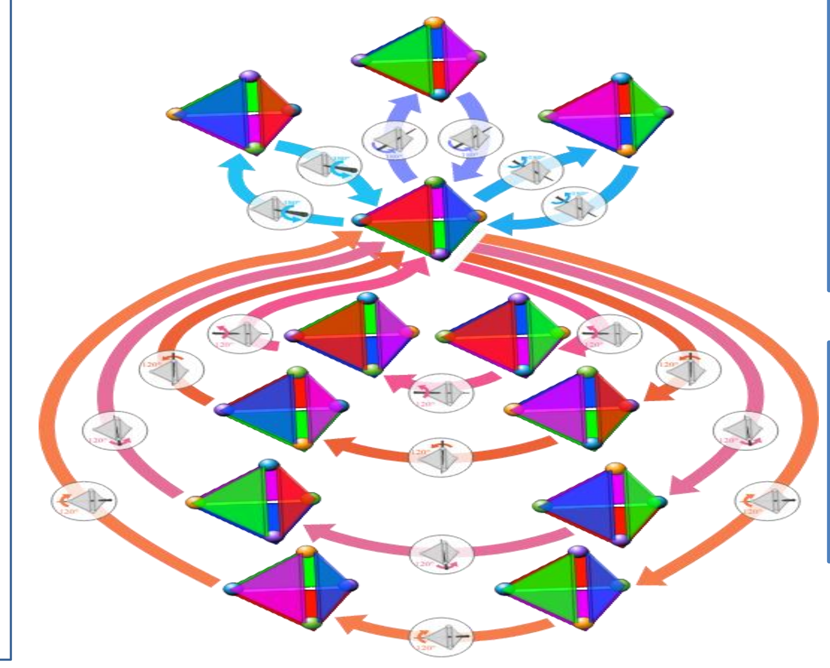
$$V = \frac{a^3}{6\sqrt{2}}$$

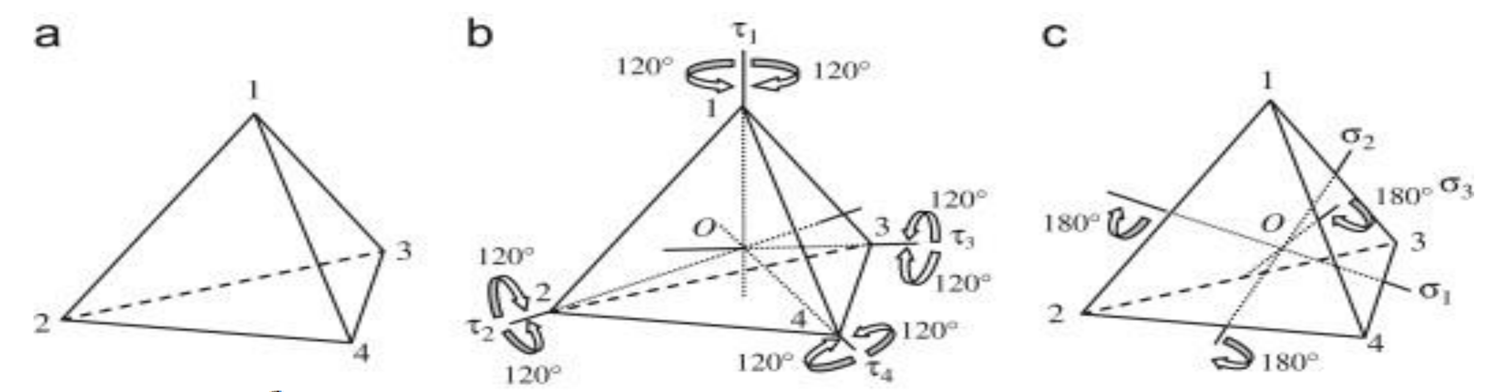
Regular tetrahedron
Solve for surface area ▾

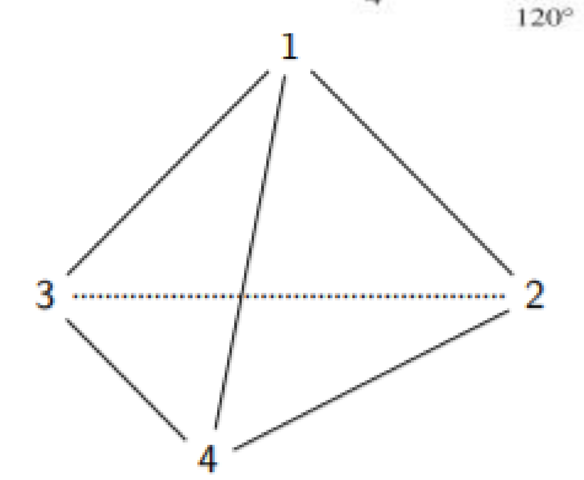
$$A = \sqrt{3}\, a^2$$

Rotations of a tetrahedron

A tetrahedron has 6 axes of symmetry, and it has 12 rotations. In the image below (Fig. X) we see a tetrahedron with vertices labelled 1,2,3 and 4. We will denote a symmetry in this set as a list (i1,i2,i3,i4) where ij is an element of {1,2,3,4} and where ij is the vertex that j is sent to via a symmetry.if we take an example of a rotation- say R1 is the rotation that fixes vertex 1 and rotates the remining vertices on the plane by 2π/3 counterclockwisse, we would write this rotation as (1,3,4,2) with a fixed vertex at 1. As written before, there are 12 rotations . A list of all 12 is given below: (Rotations are labelled R1 meaning rotation with fixed vertex 1")





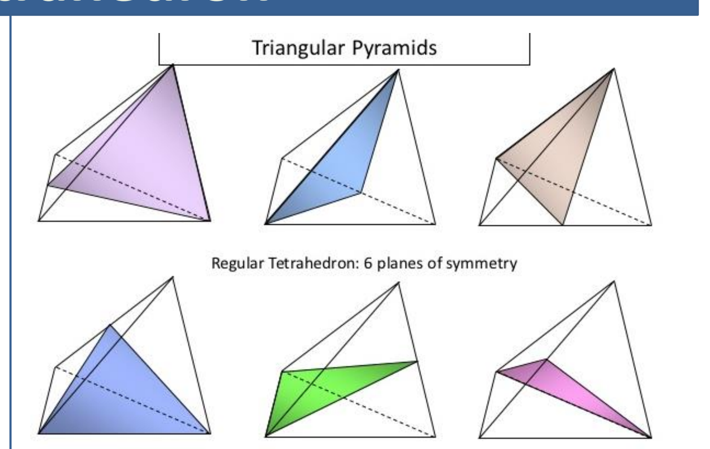


Rotations:

ID: (1,2,3,,4)	R: (2,1,4,3)	R: (3,4,1,2)
R: (4,3,2,1)	R1: (1,3,4,2)	R1: (1,4,2,3)
R2: (4,2,1,3)	R2: (3,2,4,1)	R3: (2,4,3,1)
R3: (4,1,3,2)	R4: (3,1,2,4)	R4: (2,3,1,4)

Reflections of a Tetrahedron

As you can see in the diagram (top right), there are 6 possible planes that can bisect a tetrahedron. Each plane represents a reflection, hence there are 6 different ways to conduct a reflection of a tetrahedron.
In the process of a reflection, the two vertices lying on the plane will become fixed points while the other two vertices will swap positions If we compose a reflection by itself, we will always get the identity element (1,2,3,4).
Another way of justifying that there are 6 reflections of a tetrahedron is, if there are 4 vertices and 2 of them must be chosen and swapped then 4 combinations of 2 will give you 6. i.e. 4C2 = 6.



Reflections

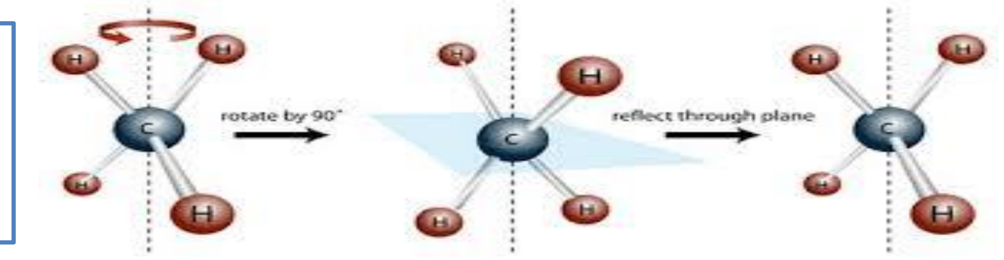
F1:(2,1,3,4)	F4:(1,3,2,4)
F2:(3,2,1,4)	F5:(1,4,3,2)
F3:(4,2,3,1)	F6:(1,2,4,3)

Rotoreflections of a tetrahedron

As we have seen so far, we have 12 rotations and 6 reflections. However, there must be 4!=24 symmetries and we have only discovered 18 of them. The remaining 6 symmetries are called rotoreflections. Rotoreflections are a composition of rotations and reflections. An example of this is in the diagram below of a rotoreflection of the organic tetrahedral compund CH_4 or otherwise known as methane. This compound goes through a rotation of 90° along with a reflection through a plane.

Rotoreflections

RF1:(3,1,4,2)	RF4:(4,3,1,2)
RF2:(3,2,1,4)	RF5:(4,3,2,1)
RF3:(2,3,4,1)	RF6:(3,4,2,1)



The subgroup S4

S4 is a symmetric group with a degree of four. It can be defined as the group of all the permutations of a symmetric group of a set size 4 {1,2,3,4}. Example of this is the symmetries of the tetrahedron.
There are 5 conjugacy classes for the symmetric group s4, they are (1+1+1+1), (2+1+1), (2+2), (3+1), (4). Each of the numbers shown here cycles, e.g. (1+1+1+1) has four cycles of size 1 and (3+1) for example is one cycle of size three with another cycle of size 1.
The group s4 contains all the symmetries of the group of the permutations of the four faces of the regular tetrahedron. As shown in the rotations, reflections and rotorelections we can see the symmetries labelled as shown in permutations to label the symmetries.

Conclusions

To conclude, This is a poster of symmetries, there are 24 symmetries of a tetrahedron, on a regular tetrahedron we can see 4 vertices, 6 edges. It is a geometrical representation of the symmetric group S4.
Our study of group theory has helped us to understand the symmetries of a regular tetrahedron, and hopefully this poster has represented the symmetries well.

References

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Examples of Companies with tetrahedral logos