The Number of Generators of a Cyclic Group

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Cyclic Groups

A cyclic group is one that is generated by a single element, in the sense that we can start with a single element and produce all the elements of G by repeatedly taking powers of that element and its inverse and by multiplying the results of such operations together.

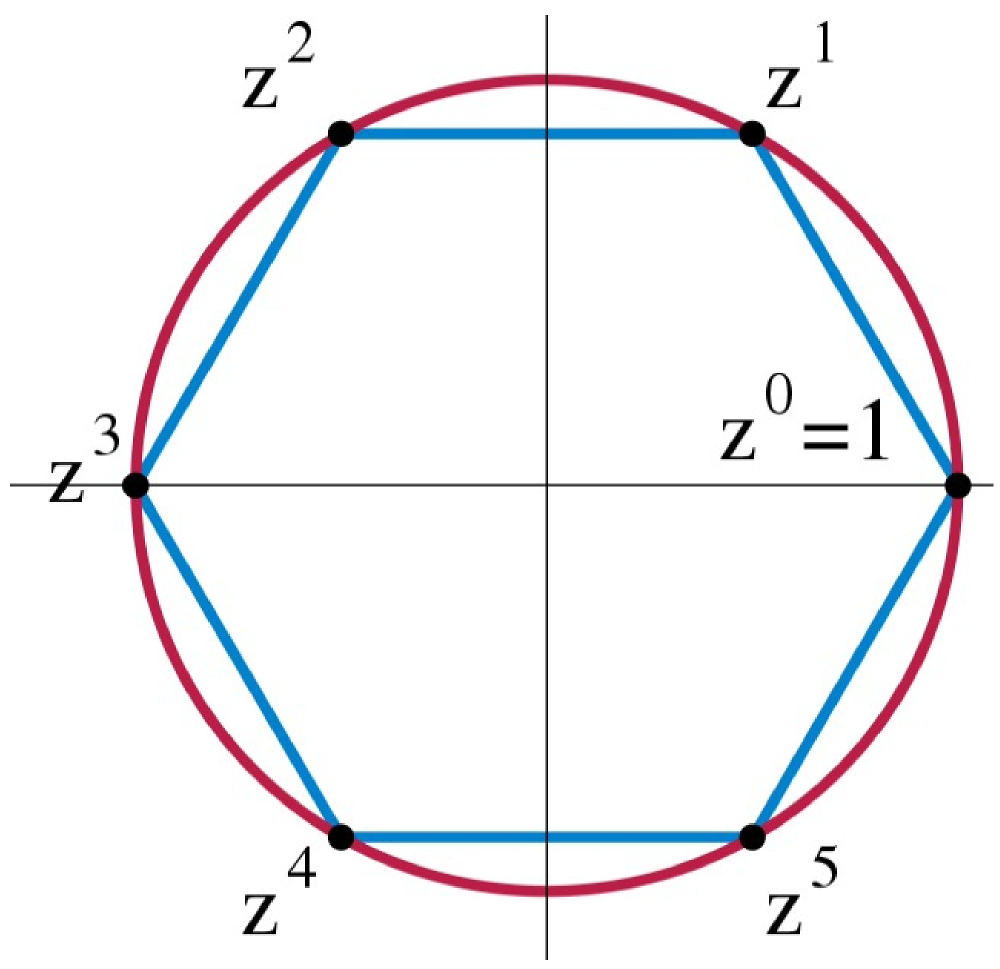
Some Examples of Cyclic Groups

- 1.) (Z, $+$) The integers under the operation of addition with 1 as a generator. This is an infinite cyclic group as all elements of the group can be acquired using $+1$ or -1. Notably the only generator in this group is $+1$ and -1.
- 2.) The Six 6^{th} Complex Roots of Unity with generator z^1, this forms a finite cyclic group under multiplication. See diagram below:
- 3.) $(\frac{Z}{nZ})$ The integers under addition modulo n, for every positive integer n. This is another finite cyclic group. If i is relatively prime to n then it is a generator of the group. A real-world example being musical notes of which there are 7 recurring notes A-G.

Relationship With The Number N

For a natural number n, $\phi(n)$ is the number of integers in the range 1, . . . , n that are relatively prime to n where ϕ is Euler's totient function. The number of generators of a cyclic group are relatively prime to the order of group. Every group of prime order is cyclic.
We can see this in the example below.
For example 12(which is Z12=(0,1,2,3,4,5,6,7,8,9,10,11)).
$\langle 0\rangle$= (0)
$\langle 1\rangle$= (1,2,3,4,5,6,7,8,9,10,11,0)= Z12
$\langle 2\rangle$= (2,4,6,8,10,0)
$\langle 3\rangle$= (3,6,9,0)
$\langle 4\rangle$=(0,4,8)
$\langle 5\rangle$= Z12
$\langle 6\rangle$= (0,6)
$\langle 7\rangle$= Z12
$\langle 8\rangle$= (8,4,0)
$\langle 9\rangle$= (9,6,3,0)
$\langle 10\rangle$= (10,8,6,4,2,0)
$\langle 11\rangle$= Z12
If its relatively prime to 12 it generates Z12. This illustrates that all generators in a cyclic group are relatively prime to that group's order.

The Six 6^{th} Complex Roots of Unity



Finite Cyclic Groups

A finite cyclic group is a group that has a finite cardinality, is cyclic and is isomorphic to the integers modulo n for some positive integer. If G is a cyclic group with a finite order n and a generator a, then G is isomorphic to $(Z_n, +)$. In other words, every finite cyclic subgroup is isomorphic to $(\frac{Z}{nZ})$, the integers mod n.
If G is finite, of order n, the group generated by g is denoted $\langle g\rangle$. The order of a finite cyclic group is the number of elements in the group, aka the cardinality of the group.
If the cyclic subgroup $\langle a\rangle$ of G is finite then the order of a is the order of the cyclic group.
Every finitely generated abelian group is a direct product of cyclic groups.

The Number of Generators in an Infinite Set

Let $G = \langle a\rangle$ be an infinite cyclic group.
Since a is the generator of G then a^{-1} is also a generator of G (as G is a group)
Let b be any generator of G. Then b and $G = \langle a\rangle$
$\rightarrow b = a^n$ for some n
For a and $G = \langle b\rangle$
$\rightarrow a = b^m$ for some m
$\rightarrow a = b^m = (a^n)^m = a^{nm}$
$\rightarrow$ nm=1 $\rightarrow m = \frac{1}{n} \rightarrow n = \pm 1 \rightarrow$ b= a or a^{-1}
$\therefore$ An infinite cyclic group, has precisely 2 generators

References

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