The Converse of Lagrange's Theorem

Megan McGlinchey and Eoin Mulvihill

Group Theory MA3343

Introduction

Lagrange's Theorem can be regarded as one of the most central theorems of abstract algebra and considered by many "the most important theorem of group theory". However, the converse of this infamous theorem is undoubtedly false. This poster will explore why the converse fails as well as other methods of finding subgroups

Joseph Lagrange

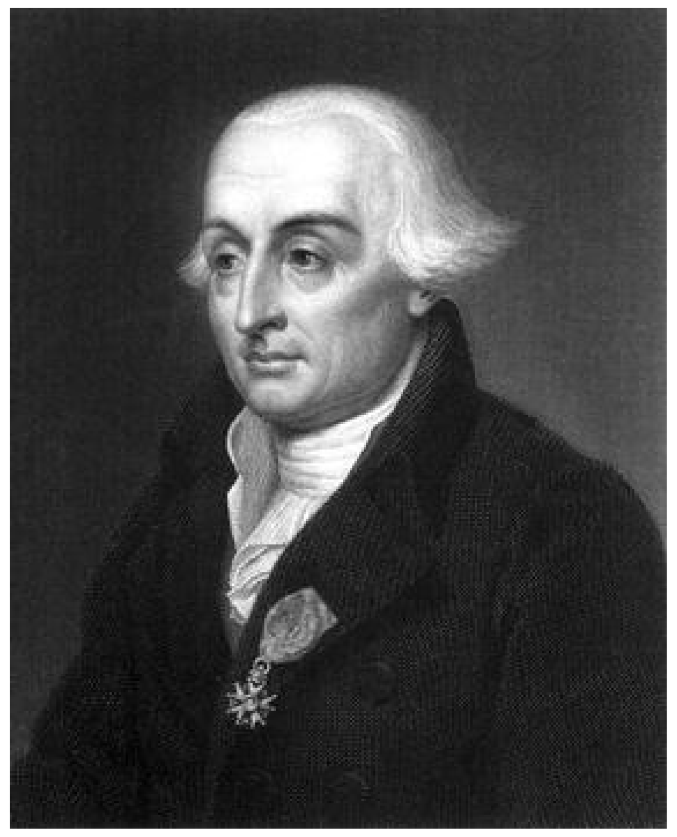

Fig. 1: Joseph Lagrange.

Giuseppe Luigi Lagrange was born in Turin, Italy on 25th January 1736. He made many significant contributions towards analysis, number theory and analytical and celestial mechanics although, sadly, did not prove his own theorem. In 1801 Gauss proved Lagrange's theorem for the multiplicative group of non-zero integers modulo p, $(\mathbb{Z}/p\mathbb{Z})$, in 1844 Cauchy proved the theorem for the symmetric group S_n and, finally, Jordan proved Lagrange's theorem for the case of any permutation group in 1861.

Applications of his work

Lagrange's Theorem displays some key properties that allow for further theorems such as Fermat's little theorem and Wilson's theorem to be proven as well as showing there to be infinitely many primes.

Converse of Lagrange's Theorem

Every divisor of the order of group G is the order of some subgroup H of G

Where the converse fails

The most basic example to demonstrate where the converse fails, is the alternating group A_4 of even permutations.

A_4={ ID,(1,2)(3,4),(1,3)(2,4),(1,4)(2,3),(123),(132),(124),(142),(134),(143),(234),(243)}

$|A_4|$=12 , With the divisors of the group being {1,2,3,4,6,12}

Lets assume there exists a subgroup H, in A_4 with the order of $|H|$=6.
Let V be a non-cyclic subgroup of A_4 –> Known as the Klein four group.
V={ID,(12)(34),(13)(24),(14)(23)}

Let $K = H \cap V$, Since H and V are subgroups of A_4, then so to is K.
By Lagrange's Theorem, K's order divides both 6 and 4. So $|K|$=1 or $|K|$=2

If $|K|$=1, the map (h, v) -> $h.v$ defined from H x V to A_4 has a one to one relationship implying A_4 has 24 elements which we know is not true.
So $|K|$ = 2 where $h \in H$ and $v \in V$

The index $|A_4 : H|$=2 shows that there is exactly 2 distinct cosets and as such, we know H is a normal subgroup.
This implies $H = tHt^{-1}\ \forall t \in A_4$.

Take $v = (ab)(cd)$ and $t = (abc)$, where (a, b, c, d)= (1,2,3,4)
tvt^{-1} = $(bc)(ad) \neq (ab)(cd)$
$tvt^{-1} \neq v$
but V contains all disjoint transpositions so,
$tvt^{-1} \in V$ and $tvt^{-1} \in H$ So, $tvt^{-1} \in H \cap V = K$. Thus, we have demonstrated that there is a third element in K which contradicts our assumption that $|K|$=2 and so there is no subgroup of order 6.

Cauchy's Theorem

Cauchy's Theorem states that a group G whose order g is divisible by a prime number p contains an operator of order p.
Proof:
Suppose G is abelian and generated by a single operator S of order np. $S^n \neq 1$ although $(S^n)^p = 1$, showing that S^n is the required operator.
If G is not generated by a single operator, we can examine a set of generating operators $\{S_1, S_2, ..., S_r\}$, which are all commutative. Since these operators are commutative there exists at least one generator for which the order is divisible by p, and some power of this generator must be the required operator of order p.

Fig. 2: Augustin-Louis Cauchy.

References

- Miller, G.A.. (1898). On an Extension of Sylow's Theorem. Bull. Amer. Math. Soc., vol. 4 p323
- https://mathshistory.st-andrews.ac.uk/Biographies/Lagrange/
- https://www.maa.org/sites/default/files/pdf/cms$_{upload}/On_{the_{C}onverse}-Gallian34078.pdf$
- https://www.mathcounterexamples.net/converse-of-lagrange-theorem-does-not-hold/
- https://en.wikipedia.org/wiki/Lagrange