Group Theory in Music: Neo-Riemannian Theory

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The Musical Space

'Music is not only a mysterious and metaphorical art; it is born of science ...it is made of mathematically measurable elements ...any explanation of music must combine mathematics with aesthetics' [3]

Group theory can be applied to the basics of music theory as a means of enriching our understanding of the machinery of musical operations, and also as a means of highlighting the real-life applications of this branch of mathematics. Firstly, however, we must address the concept of the musical space.

Assuming octave equivalence (i.e. a given C note is equivalent to all C notes in higher and lower octaves) and enharmonic equivalence (i.e. $C\sharp = D\flat$) we can assign an integer modulo 12 to every note in the musical space as in Fig.1 below.

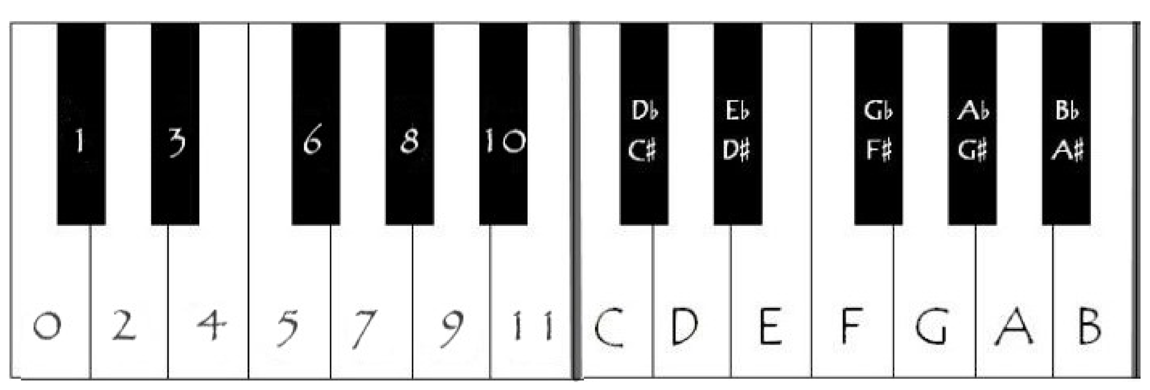

Fig. 1: Pitch-Class Integers [5]

We can then think of the interval between any two notes a, b as

$$a - b \mod 12$$

Furthermore, each major and minor chord can be represented by a pitch-class set consisting of the three pitch-class integers that form the triad. For example, a C major triad is denoted by (0,4,7).

Transpositions as Cyclic Groups

A transposition $T_n(x)$ in the musical space moves a pitch $x \in \mathbb{Z}_{12}$ up by $n \mod 12$. If we take $T_7(x)$ to be the transposition that moves each x up by $7 \mod 12$, we can generate the entire group of all such transpositions [4].

$$< T_7 >= \{T_7, T_2, T_9, T_4, T_{11}, T_6, T_1, T_8, T_3, T_{10}, T_5, T_0\}$$

This represents a circle of fifths starting at the pitch-class x. If $x = 0$, the pitch-class of C, then:

$$< T_7(C) >= \{G, D, A, E, B, F\sharp, C\sharp, G\sharp, E\flat, B\flat, F, C\}$$

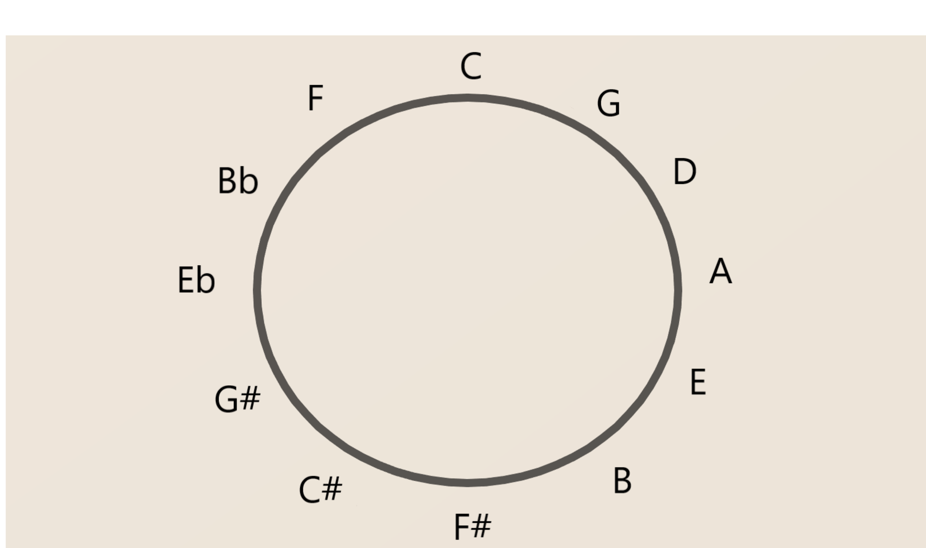

Fig. 2: Circle of Fifths generated by $T_7(x)$

Inversions & the TI Group

It is worth noting that the group of transpositions mentioned earlier can be applied to single notes, triads or to melodic phrases. If we define an inversion I_n of a triad $x = (a, b, c)$ as

$$I_n = -x + n = (-a + n, -b + n, -c + n)$$

and then form the set of all transpositions and inversions, defined as

$$TI = \{T_n, I_n : n = 0, ..., 11\}.$$

It is shown in [2] that TI is a group under composition:

1. Composition of functions is always associative by definition.
2. It can be shown that TI is closed by considering

$$T_m \circ T_n = T_{m+n \mod 12}$$
$$T_m \circ I_n = I_{m+n \mod 12}$$
$$I_m \circ T_n = I_{m-n \mod 12}$$
$$I_m \circ I_n = T_{m-n \mod 12}$$

3. Note that $id_{TI} = T_0$ as

$$T_0 \circ T_n = Tn = T_n \circ T_0$$
$$I_n \circ T_0 = I_n = T_0 \circ I_n$$

4. By the equations in 2, the inverse of each T_n is T_{12-n} and the inverse of each I_n is itself.

PLR Group

Although the TI group can be useful in terms of showing that we can get from any triad to another simply by applying transpositions and inversions, the TI group falls short in its musical practicality. As well as possessing a practical musical description, the set of PLR transformations as defined below also has a strong connection with group theory. The PLR group forms a key part of Neo-Riemannian theory, put forward initially by David Lewin and based largely on the work of Hugo Riemann. Each of the PLR operations can be defined as follows:

P: The parallel operation P sends a triad to its unique triad of opposite parity. For example, P sends C major=(0,4,7) to C minor=(0, 3, 7) and vice versa.

L: The leading-tone exchange operation L shifts the bottom note of a major triad down by a semitone and shifts the top note of a minor triad up by a semitone.

$$L(Cmajor) = L(0, 4, 7) = (11, 4, 7) = Eminor$$
$$L(Cminor) = L(0, 3, 7) = (0, 3, 8) = A\flat major$$

R: The relative operation R sends a major triad to its relative minor by shifting the top note of the triad up a whole tone (up by 2). Conversely, R sends a minor triad to its relative major by shifting the bottom note down by a whole tone.

$$R(Cmajor) = R(0, 4, 7) = (0, 4, 9) = Aminor$$
$$R(Cminor) = R(0, 3, 7) = (10, 3, 7) = E\flat major$$

It is woth noting that P, L, R are involutive:

$$P^2 = L^2 = R^2 = id$$

PLR Group = D_{24}

It is proved in [2] that the set of all PLR transformations forms a group under composition. It can also be shown that this PLR group is equivalent to D_{24}, the dihedral group of order 24.

D_{24} is generated by a rotation r and a refelction s such that

$$s^2 = r^{12} = id_{D_{24}}$$
$$srs = r^{-1}$$

We can see that the PLR group consists of 24 distinct bijections and that the group has order of at least 24, by alternately applying R and L to the C major triad:

$$C, a, F, d, B, g, E, c, A, f, D, b, G, e, B, g, E, c, A, f, D, b, G, e, C$$

where capital letters represent major triads and lower case letters represent minor triads. Note that $R(LR)^3 = P$, which tells us that the PLR group can be generated by L and R.

Now let $r = LR$ and $s = L$. Then $r^{12} = id = s^2$ and

$$srs = L(LR)L = RL = s^{-1}$$

Finally, [1] shows that the PLR group has exactly 24 elements by showing it is a subgroup of the aforementioned TI group which has order 24.

The Tonnetz

The *Tonnetz*, or tone network, can be used as a geometric representation of the PLR transformations. Since Euler first intouduced it, the tonnetz has been developed by Hugo Riemann and David Lewin, among others. As the tonnetz in Fig.3 is expanded, it repeats and can be then projected onto a torus. Note that each triangle in Fig.3 represents a major or minor triad which together, triangulate the torus. Also, since each PLR action only alters one note, these actions can be considered as reflections of triangles about one of its edges.

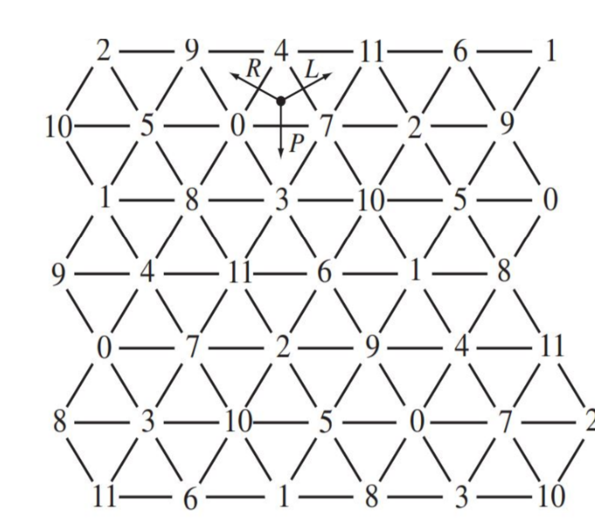

Fig. 3: Tonnetz [2]

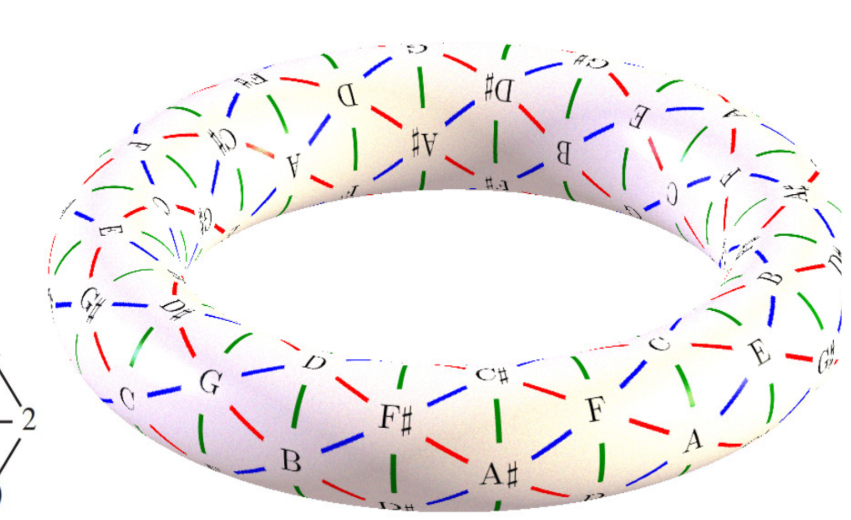

Fig. 4: Tonnetz lying on a torus

References

[1] Alissa S. Crans et al. "Musical Actions of Dihedral Groups". In: *American Mathematical Monthly* 116 (2007), pp. 479–495.

[2] F. Aceff-Sánchez et al. *An Introduction to Group Theory with applications to Mathematical Music Theory.* Publicaciones Electrónicas Sociedad Matemática Mexicana, 2012.

[3] L. Bernstein. *The Unanswered Question.* Harvard University Press, 1976.

[4] E.B. Roon. *That strikes a chord! An illustration of permutation groups in music theory.* 2019.

[5] A. Zhang. *The Framework of Music Theory as Represented with Groups.*