LAGRANGE'S THEORM

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MA3343 Groups Poster

Introduction

In group theory, there is a well-known theorem which defines the correlation between the order of a group and the order of its subgroup. This theorem is called Lagrange's Theorem named after the Italian mathematician Joseph Louis Lagrange. This poster will explore the man himself, his theorem and its evolution into what we know Lagrange's Theorem as today.

What did Lagrange do?

Lagrange's was concerned with the question of finding an algebraic formula for the roots of the general 5_th degree polynomial and more generally for the nth (n > 4), since the quadratic, cubic and quartic formulae were already known. He observed that to solve the quartic and cubic equations involved solving supplementary polynomials of lower degree, whose coefficients were rational functions of the coefficients of the original polynomial. These polynomials are also known as "resolvent"([2]) polynomials. For this example we can write the roots as:

$$\frac{x1x2 + x3x4}{2}$$

,

$$\frac{x1x3 + x2x4}{2}$$

,

$$\frac{x1x4 + x2x3}{2}$$

where x1, x2, x3, x4 are roots of the original polynomial. Lagrange also observed that all four roots could be permuted in 4! = 24 possible ways and only these three values would occur. He then concluded: To solve 5_th degree polynomials one should try to find function in 5 variables that takes on 3(or 4) different typical values when the variables are permuted in all 5! ways.

Example: Lagrange's Theorem for C_6

We can illustrate the key ideas behind the proof of Lagrange's Theorem using the example of C_6 = (1, g, g$^2, g^3, g^4, g^5)(where g^6 = 1)$
which has as one of its subgroups H = (1, g^3).
If we multiply H on the right by each element of C_6 in turn we find the different right cosets of H in G. (1, g^3), (g, g^4), (g^2, g^5)
Using some color we can see how
H = (1, g^3)
gets "shifted" through the elements of C_6 as we multiply H on the right by elements of C_6.
C_6 = (1,g, g^2, g^3, g^4, g^5)
There are three distinct cosets.

1. Each right coset has the same size as H.
2. Two cosets are either equal or disjoint.
3. Every element of G is in exactly one right coset.

These are the key ideas needed to prove Lagrange's Theorem.

Joseph Louis Lagrange

Joseph Louis Lagrange (Giuseppe Luigi Lagrangia) was born in Turin, Italy on the 25th of January 1736. He made many contributions to several areas of mathematics, including mechanics, analysis and number theory. He is renowned in mechanics, where he reformulated Newtonian mechanics in order to simplify formulae and calculations, this is known as Lagrangian mechanics. His name is also associated with the Euler-Lagrange equation in calculus. In 1766, Lagrange moved to Berlin to pursuit mathematics after studying in the University of Turin. In 1766 he started to work on what became to be known as Lagrange's Theorem. He passed away in Paris in 1813, aged 77.

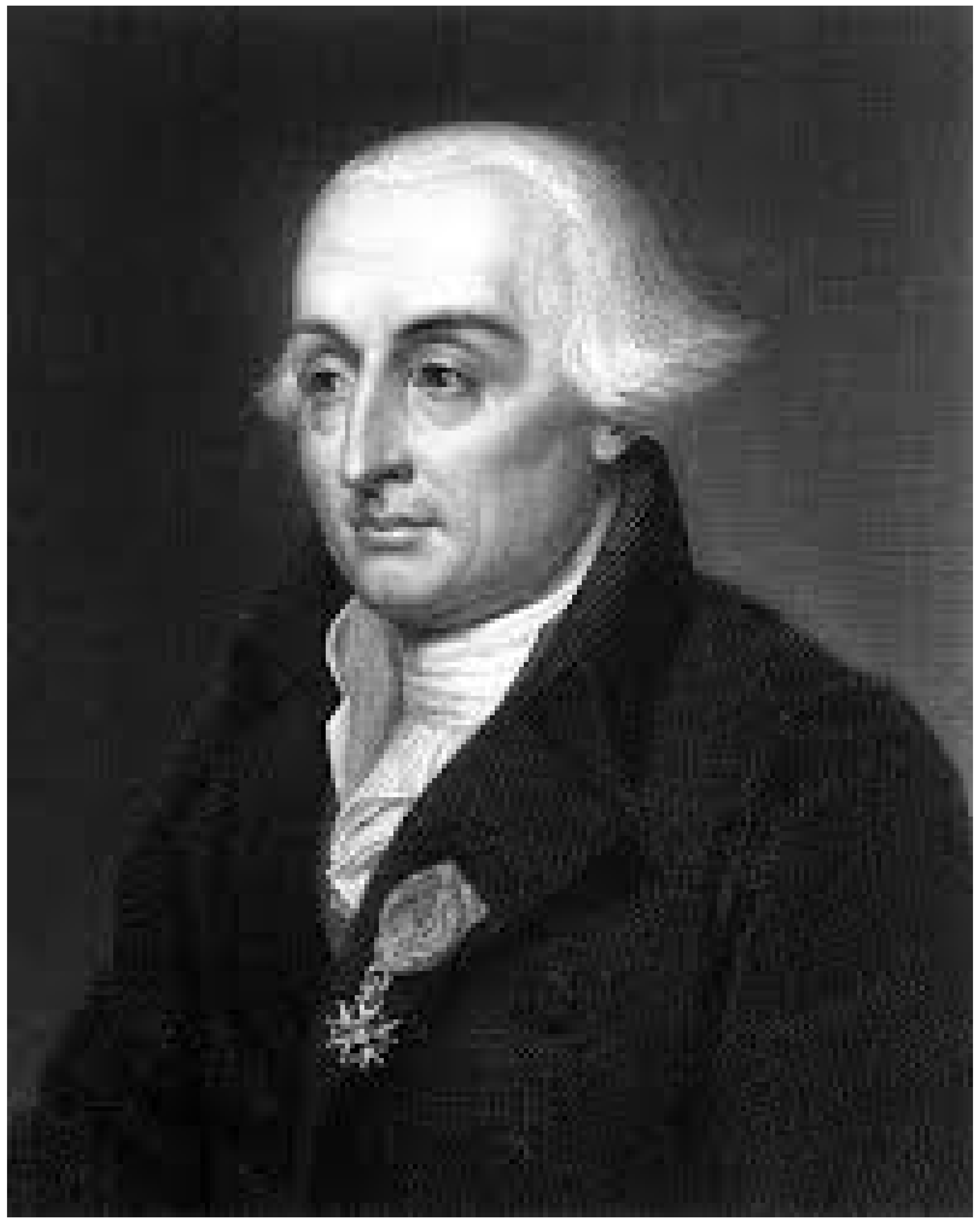

Fig. 1: joseph lagrange.jpg

Key Achievements

•Built on earlier work by Leonhard Euler to create the calculus of variations – he called it his 'method of variations.'
•Created an entirely new field of mechanics, Lagrangian mechanics, for both solids and fluids, based on the concept of virtual work and utilizing the Lagrangian function.
•Introduced the concept of generalized coordinates. Lagrangian mechanics can be used in any coordinate system – problems are simplified by choosing an appropriate one.
•Created the concept of potential: the gravitational field, for example, is a potential field.
•Discovered Lagrangian orbits.
•Solved century-old problems in number theory posed by Fermat that had defeated other mathematicians.
•Was a founder of group theory.
•Played a key role in the creation of the metric system of weights and measures.

Lagrange's Theorem

Find the subgroups that some finite group G possesses can be quite difficult. However, Lagrange's Theorem gives us more knowledge on how to find those subgroups, making the process much easier.

Theorem

If G is a finite group and H is a subgroup of G, then |H|, the order of H divides |G|, the order of G. This proves very helpful in figuring out which subgroups a group possesses provided the group is finite.For example, it tells us that a group G of order 24 cannot possibly have a subgroup of order 11.

References

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