Group Theory applied to the Rubik's Cube

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Introduction

In our 3rd year project of Group Theory, we decided to see how all the information we learned in this module applies to the Rubik's Cube, a puzzle designed by the teacher and puzzler Erno Rubik in 1974.

Proof the Permutations of a Rubik's Cube are a Group

The 4 axioms that must be fulfilled in order to be a group are closure, associativity, identity and inverse.

- Closed Property : If A and B are operations then A*B is an operations as well.
- Associativity : For all A, B, C in G, one has (A*B)*C = A*(B*C))
- Identity Property : The identity id. can be defined as the move that does not perform any rotations and remains as is.
- Inverse Property : For each A in G, there exists an element B in G such that A*B = id., and B*A = id., where id. is the identity element.

Notation of the Rubik's cube

Generally, the actions on a Rubik's cube are notated as follows:

F - rotate the front side of the cube 90° clockwise
B - rotate the back side of the cube 90° clockwise
L - rotate the left side of the cube 90° clockwise
R - rotate the right side of the cube 90° clockwise
U - rotate the top side of the cube 90° clockwise
D - rotate the bottom side of the cube 90° clockwise

Placing a ' after the letter implies you rotate the face *anticlockwise* instead e.g. F' would mean rotate the front face anticlockwise.

Placing a '2' after the letter implies rotation by 180° instead. e.g. B2 means rotate the back face 180°.

Subgroups of the Rubik's Cube

A *subgroup* of a group is simply a subset of the group that also satisfies the group axioms. (closed, inverse, etc.)

For example, define S to be:

$$S := \{id, L, L2, L'\}$$

Then S is a subgroup of the whole group.

- S is clearly a subset of the group
- S contains the group identity
- The composition of 2 elements in S is also in S (e.g. L*L2 = L')
- Every element in S can be undone by another element of S (or in the case of L2, itself)

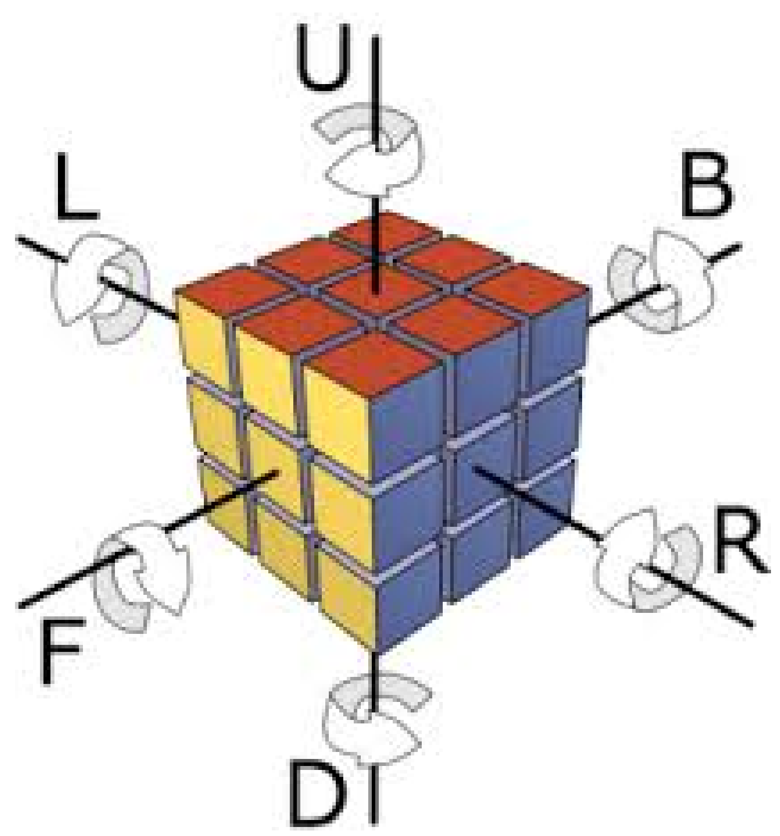
Commutativity and Centre of the Rubik's Cube

A group is *commutative* if a*b = b*a for all a, b $\in G$. Such groups are known as *abelian*.

The Rubik's Cube Group is not abelian. If we rotate our L axis and then our F axis both in a clockwise direction, we cannot return to our original position just by repeating those moves in reverse order.

The *centre* of a group is the set of elements that commute with EVERY element in the group. In our Rubik's Cube, the centre of the group consists only of our identity element and a move known as the "Super Flip"

The Notations of the Rubik's Cube



The Super Flip

The Super Flip of any Rubik's Cube is a series of 20 moves that, when completed, transforms the cube so that the corners and centre of each face remain in the same position, but the edge pieces ("cubies") are flipped. If we were to complete the Super Flip twice, we get our original cube (The Super Flip has order 2).

Super Flip: U*R2*F*B*R*B2*R*U2*
L*B2*R*U'*D'*R2*F*R'*L*B2*U2*F2

Orbits of the Rubik's Cube

The *orbit* of a point (e.g. face, corner, or even one of the cubies) of the cube under a group is every position that point can be in under the action of the group.

For example, each of the corner cube form an orbit under the whole group, as each corner cube can only be moved to another corner. The order of this orbit is 8 as there are 8 corners.

Similarly, the "cubies" form an orbit of order 12.

Finally, the centre squares don't change position (apart from rotating, but this is irrelevant), so its orbit size is just 1.

Stabilizers of the Rubik's Cube

The *stabilizer* of a point under a group is every element of the group that keeps that point exactly where it is.

For example, performing any of B, B2, or B' wont move any of the faces on the front side of the cube, so we say B, B2 and B' are in the stabilizer of these faces.

References

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http://people.math.harvard.edu/ jjchen/docs/rubik.pdf