The Exponent of Groups

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MA3343 Poster Project

Our Objectives

- To clearly explain the concept of an Exponent and its relevance within Group Theory.
- To show the exponent of a finite group is the LCM of the orders of the elements of the group.
- To deduce that the exponent of a finite group is a divisor of the order.
- To determine the exponent of the dihedral group of order 2n.
- To establish properties of groups with different exponents.

The Concept

Let G be a group and let k be a positive integer such that $x^k = id \ \forall \ x \in G$.

The exponent of a group G is the **least** $k \in \mathbb{Z}^+$ with the property that $x^k = id \ \forall \ x \in G$. If no such k exists then the exponent is infinite.

A finite group **always** has a finite exponent since all elements have finite orders.

We can define the exponent of a group G to be the **least-common multiple** of the orders of each element of the group:

- Given the order o_i of an element $x_i \in G$ with $x_i^{o_i} = id \ \forall \ x_i \in G$, The exponent of G is

$$LCM(o_1, o_2, o_3, \ldots, o_n).$$

Lemma

The exponent of a group divides the order of a group.

Given a group G, let the order of G, $|G| = Y$. Let the exponent of G, $LCM(o_1, o_2, \ldots, o_n) = E$. We need to show that $E|Y$.

By **Lagrange's theorem**, we can deduce that the order of an element, $x_i = o_i$, is a **divisor** of the group order: i.e. $o_i|Y \ \forall \ x_i \in G$. We also have that $x_i^Y = id \ \forall \ x_i \in G$.

We defined E as the smallest k such that $x_i^k = id \ \forall \ x_i \in G$.

In addition, $x_i^{cE} = id \ \forall \ x_i \in G$ since $x_i^{cE} = (x_i^E)^c = id^c = id \ \forall \ x_i \in G, \ c \in \mathbb{Z}$. So, $Y = cE$ or $E|Y$.

Lemma

p is a prime factor of $E \iff p$ is a prime factor of $Y = |G|$.

Let p be a prime factor of E. Since $E|Y$, any factor of E must be a factor of Y, and so p is a prime factor of Y.

Conversely, let p be a prime factor of Y.

Cauchy's Theorem: if p divides the order of a finite group G, then $\exists$ an element of G with order exactly p. By definition of the exponent of a group, E is a multiple of the order of all elements in G. So $E = mp$, for some $m \in \mathbb{Z}$. So p is a prime factor of E.

D_{2n} : The Dihedral Group of order 2n

We can apply the definition of the exponent of a group G to D_{2n}, the group of symmetries of a regular n-gon, for $n \geq 3$.

Reflections: We know that $|T_p|$, where T_p is any reflection is 2, since $T_p^2 = id$.

Rotations: Since every rotation R_i belongs to $\langle R_{\frac{360}{n}} \rangle$, and so $|R_i|$ divides $|R_{\frac{360}{n}}|$, where $|R_{\frac{360}{n}}| = n$. So $lcm(o_{R_i}) = n$.

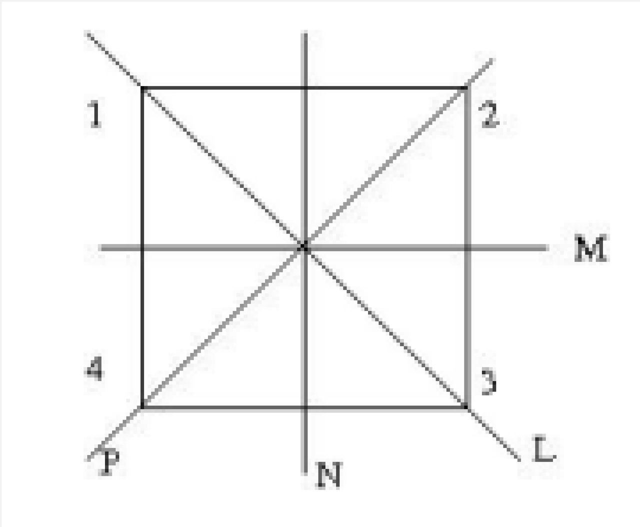
Therefore, the exponent of any $D_{2n} = lcm(2, n)$, i.e. n, if n even, or $2n$, if n odd.

Example: Exponent of D_8 : Group of Symmetries of the Square

$D_8 = \{id, R_{90}, R_{180}, R_{270}, T_L, T_M, T_N, T_P\}$ with respective orders 1, 4, 2, 4, 2, 2, 2, 2.

The exponent of $D_8 = LCM(1, 2, 4) = 4$.

Above, we showed that the exponent of any $D_{2n} = LCM(2, n)$, so for D_8 we have $E = LCM(2, 4) = 4$, which agrees with our manual calculation.



Some Facts

- Groups with exponent 1 are **trivial**.
- Groups with exponent 2 are **abelian**.
- Groups with exponent 3 are not necessarily abelian.
- Every abelian group of n elements have exponent = $MAX(o_1, o_2, \ldots, o_n)$.
- If G is a finite abelian group and $E = |G|$, then G is **cyclic**.

Application of Group Exponent

Burnside Problem: Must a finitely generated group in which every element has finite order be a finite group? The question became: Are all finitely generated groups of bounded exponent finite? Another question answered by Russian mathematician Zelmanov was: For fixed m, n $\in \mathbb{Z}^+$, are there only finitely many groups generated by m elements of bounded exponent n?

References

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