The History of Lagrange

Emer Forde, Fionnuala Forkan, Eoghan Gallagher

National University of Ireland - Galway

Introduction

Joseph Louis Lagrange was a mathematician in the 18th century, He discovered and proved Lagrange's Theorem. This theorem is seen in the foundations of Group Theory.

This poster will give you an insight to Lagrange's life, the development and proof of his Theorem and uses of his thoerem within Group Theory.

How did Lagrange Discover His Theorem

Linear, Quadratic, Cubic and Quartic equations can be solved using an algebraic formula. Lagrange wanted to find a solution to find the roots of Quintic or higher degree equations. He investigated previous solutions to cubic and quartic equations. He took a different approach to these equations by considering them in terms of permutations of roots.
This was the beginning of group theory before the idea of group theory existed.

He found that cubic and quartic equations can yield an auxillary equation of lower degree (known as a resolvent) which can be used to find the roots of the original polynomial. 'The quartic was solved using a cubic resolvent polynomial whose roots could be written as

$$\frac{x_1x_2+x_3x_4}{2}, \frac{x_1x_3+x_2x_4}{2}, \frac{x_1x_4+x_2x_3}{2} \text{ where } x_1, x_2, x_3, x_4$$

Were the roots of the original polynomial.'[1] These roots can be permuted in every way 4! = 24 and give 3 values. He tried to do this with the quintic equation, which has 5 variables which gives 5! = 120 permutations. He wanted to find an equation that would give either 3 or 4 values within all 120 permutations.

Lagrange did not succeed in this could not find a general formula to solve the quintic and it was later proven to be impossible in the Abel–Ruffini Theorem. However Lagrange did discover his Thoerem during his attempt.

Lagranges Original Thoerem

Lagrange's Original Theorem is not what we see in Group Theory today. In his article Réflexions sur la résolution algébrique des équations, Lagrange states:
'*If a function $f(x_1,...,x_n)$ of n variables is acted on by all $n!$ possible permutations of the variables and these permuted functions take on only r distinct values, then r is a divisor of $n!$.*'[1]
This theorem has evolved into a Theorem that can be used in what we call Group Theory


LEÇONS

RÉSOLUTION ALGÉBRIQUE

DES ÉQUATIONS

Avec une Préface

M. JULES TANNERY

PARIS

LIBRAIRIE NONY & Cie

Fig. 1: Réflexions sur la résolution algébrique des équations

Who was Lagrange

- Joseph Louis Lagrange was born on 25th January 1736 and died in Paris on 10th April 1813. He was an Italian Mathematician and Astronomer. He made significant contributions to the fields of analysis, number theory and mechanics.
- He was educated at the College of Turin where he found a keen interest for mathematics at the age of 17, after reading a memoir by Edmond Halley on the use of algebra in optics.
- At 19 years of age he wrote a letter to Euler in which he solved the isoperimetric problem. The paper draws an analogy between the binomial theorem and the successive derivatives of the product of functions.
- Euler recognized the generality of this methods and its superiority to his own method. This placed Lagrange in the front rank of mathematics of that time.
- On 18 May 1787 he left Berlin after 20 years, to become a member of the Académie des Sciences in Paris, where he remained for the rest of his career.
- Famous Lagrange Quote: "As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company, they drew from each other fresh vitality and thenceforward marched on at a rapid pace toward perfection."

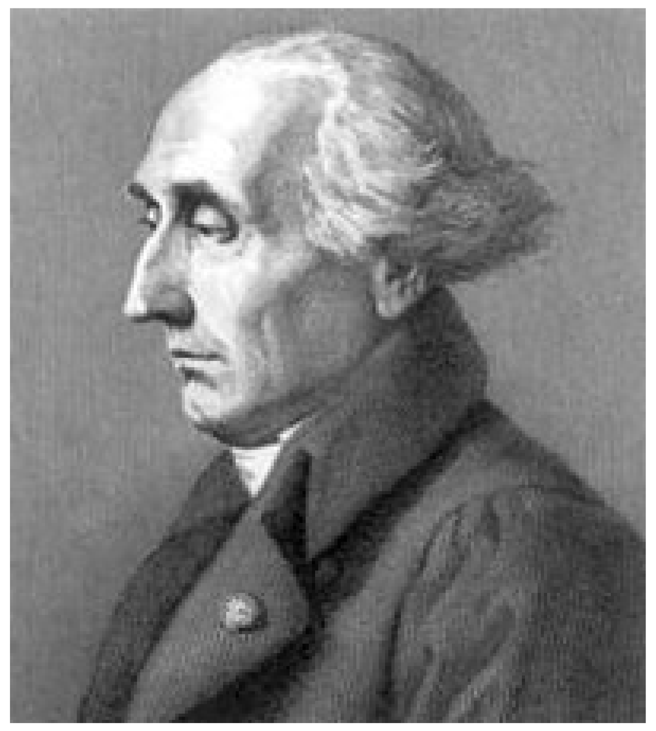

Fig. 2: Lagrange

Lagranges Thoerem

For a finite group G, the order of any subgroup H divides the order of G
Therefore, the order of H is a factor of the order of G.
(order = no. of elements).

Proof

Relies on three lemmas:
- If G is a group with subgroup H then there is a one-to-one correspondence between H and any coset of H
- If G is a group with subgroup H then the left coset relation $g_1 \sim g_2$ if and only if $g_1 * h = g_2 * H$ is an equivalence relation.
- Let S be a set and $\sim$ be an equivalence relation on S. If A and B are two equivalence classes with $A \cap B \neq 0$ then $A = B$

Let $\sim$ be the left coset equivalence relation defined in Lemma 2. It follows from Lemma 2 that $\sim$ is an equivalence relation and by Lemma 3 that any two distinct cosets of $\sim$ are disjoint.
Hence, $G = (g_1 * H) \cup (g_2 * H) \cup \ldots \cup (g_l * H)$
Where $g_i * H, \quad i = 1, 2, \ldots, l$ guaranteed by lemma 3.
By lemma 1, the cardinality of each of these cosets is the same as the order of H and so
$|G| = |g_1 * H| + |g_2 * H| + \ldots + |g * H| = |H| + |H| + \ldots + |H|$
L summands $= l * H = l * k$

Lagranges Theorem in C6

Lagranges Theorem can be seen in action for the group of 6 the roots of unity which has all proper subgroups of order 2 or 3 a factor of 6.

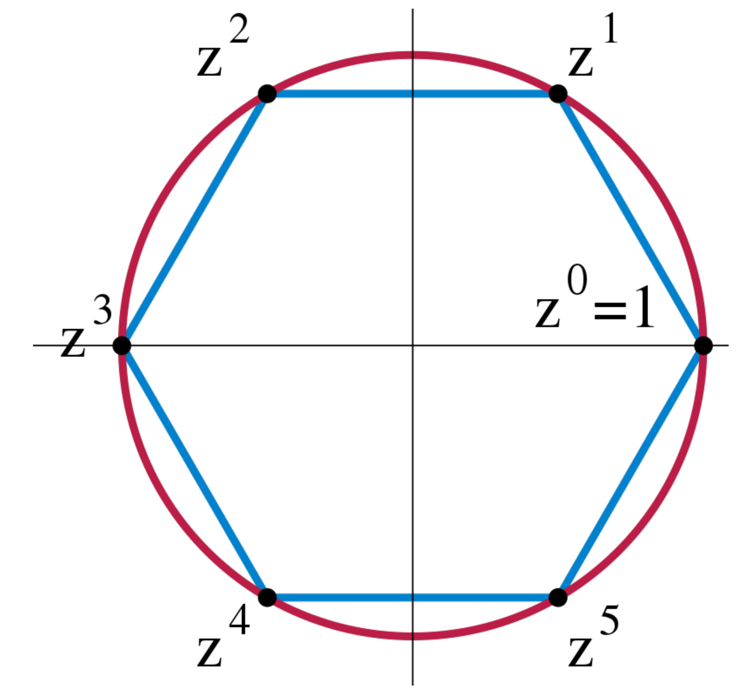

Fig. 3: Cyclic group c6.

Interesting facts about Lagrange

- Lagrange's parents were Italian, although he also had French ancestors on his father's side. In 1787, at age 51, he moved from Berlin to France and became a member of the French Academy, and he remained in France until the end of his life. Therefore, Lagrange is alternatively considered a French and an Italian scientist.
- In 1766 Lagrange succeeded Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin.
- Lagrange's treatise on analytical mechanics, first published in 1788, was the best treatment of classical mechanics since Newton, and helped the development of mathematical physics in the nineteenth century.

Euler - Lagrange Theorem

In the calculus of variations, the Euler-Lagrange equation is a second-order partial differential equation whose solutions are the functions for which a given functional is stationary. This was one of Lagranges large contributions to Mechanics

$$I(x) = \int_b^a F(x(t), x0(t), t)dt$$

References

[1] Richard L. Roth. "A History of Lagrange's Theorem on Groups". In: *Mathematics Magazine* 74.2 (2001), pp. 99–108. ISSN: 0025570X, 19300980. URL: http://www.jstor.org/stable/2690624.