Rubik's Cubes & Group Theory

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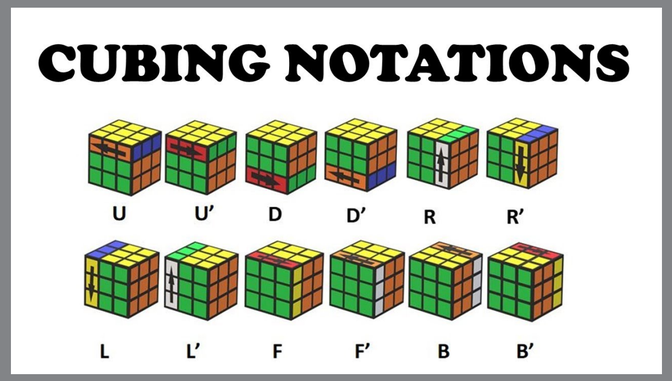
RUBIK'S CUBE

The Rubik's Cube is a 3-D combination puzzle invented in 1974 by Hungarian sculptor and professor of architecture Ernő Rubik. Although it reached it's peak popularity in the 1980's it is still widely know and used today.
Many "speedcubers" still use it and continue to improve on solve times.
The current world record for solving a cube is 3.47 seconds and is held by Yusheng Du.

How Group Theory and Rubik's Cube's Relate

All the possible rotations of a Rubik's Cube can be proven to be a group. First we need some notation for these rotations. Let $(R,\star)$ denote the group, where R is the set of all possible rotations, and $R_1 \star R_2$ is defined as rotating by R_1 then by R_2. Let G represent the Rubik's Cube.



U: Up, D: Down, L: Left, R: Right, F: Front, B: Back & ' is the inverse.

Proof that a Rubik's Cube is a group

Identity Element
Let R be any rotation, Let e be the Identity. Then
$e \star R = R$, $R \star e = R$
Therefore e is the Identity.

Closure
Let R_1 & $R_2 \in$ G.
For any $R_1 \star R_2$ it will produce a valid move, therefore $R_1 \star R2 \in G$. Thus is closed.

Inverse
Let R be any rotation in G, Let R' be the inverse to R. So $R \star R' = e$ and $R' \star R = e$, this shows that every rotation R has an inverse in G.

Associativity
Let R_1, R_2, $R_3 \in$ G.
$R_1 \star (R_2 \star R_3) = (R_1 \star R_2) \star R_3$
This is clearly true as for example, If you do the move $R \star (R \star U)$ its the exact same as $(R \star R) \star U$.
Therefore its associative and thus is a group.

How Group Theory Helps with Solving Rubik's Cubes

Group theory helps with solving cubes is by helping us find algorithms to solve them, as there is a very large number of permutations it is impossible to solve randomly so we need these algorithms.
These are made by looking at the commutativity of the group. The Rubik's Cube group is non-abelian ie, does not always commute
This property helps with solving them tremendously as we can devise algorithms to swap certain "cubies" (these are what the individual "cubes" are called), being able to swap these like this and not move the rest of the cubies is fundamental to solving a Rubik's cube.

Another way groups helps with solving a cube is with conjugates.
If R_1 and R_2 are two moves then the conjugate of R_1 is equal to $R_2 R_1 R'_2$
The conjugate has the same function as the original move R_1 but does the move in a different location.
This is very useful for cycling through cubies on an edge for example if you wanted to just move the top 3 corners a conjugate would be very useful here.

Other fact's

Number of permutations
The number of permutations possible for a Rubik's cube is a very large number.
First we need to take the 8 corner pieces so they can be arranged in 8! ways.
Then each corner piece can be arranged in 3^8 ways.
There are 12 edge pieces which can be arranged in 12! ways, then each of them has 2^{12} ways to be arranged.
But only $\frac{1}{3}$ of the permutations have the rotations of the corner cubies correct.
Only $\frac{1}{2}$ of the permutations have the same edge-flipping orientation as the original cube, and only $\frac{1}{2}$ of these have the correct cubie-rearrangement parity. So we end up with

$$\frac{(8!)(3^8)(12!)(2^{12})}{(2)(2)(3)} = 4.3x10^{19}$$

permutations.

God's Number
What is God's number, its the minimum amount of moves necessary to solve any Rubik's Cube from any state.
First we need to make a distinction between the half turn metric and the quarter turn metric.
A half turn is where any turn of 90, 180 or 270 degrees is one move Whereas a quarter turn any twist of the face is said to be a move.
God's Number is exactly 20 for the half turn metric, this was proved by Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge in 2010. It's 26 for the quarter turn metric, this was proved by Tomas Rokicki and Morley Davidson in 2014.
The superflip is the first position proven to require 20 moves (or 26 depending on which metric you prefer) the superflip is the position where all the corners are correct but all the edge pieces are flipped, the superflip actually commutes with every possible move.