THE WORLD OF ABELIAN GROUPS

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A look at commutativity

To discuss Abelian Groups, we first have to look at **commutativity.** When something is commutative, it means that the order in which we operate on them (this can be addition, multiplication etc.) does not affect the outcome. For example, addition over the Real numbers is commutative. It does not matter which order you add any amount of numbers in, the answer will be the same. We can also say this for multiplication.

$$a + b = b + a$$

It is important to realise that it is not the object that is being operated on that is at the root of commutativity, but it is in fact the operation itself. Note that not every operation on the Real numbers is in fact commutative, like division, for example. The root of commutativity is actually what is done to the objects (the operation) rather than what the objects are.

What is an Abelian Group?

In mathematics, an abelian group, also called a commutative group, is a group in which the result of applying the group operation to two group elements does not depend on the order in which they are written. That is, the group operation is commutative.

With addition as an operation, the integers and the real numbers form abelian groups, and the concept of an abelian group may be viewed as a generalization of these examples. Abelian groups are named after early 19th century mathematician Niels Henrik Abel.

Fig. 1: Norwegian Mathematician **Niels Henrik Abel**

The concept of an abelian group is fundamental to group theory. We see it appear in many areas of the subject. One of the first visual representations we get is in the form of a multiplication table. For abelian groups, their multiplication tables are symmetric.

(Note: Abelian Groups can either be finite or infinite.)

Facts and Implications

There are certainly many implications that arise from a group being abelian. Being able to define and interpret an abelian group, whilst straightforward, can prove very useful.

1. Every subgroup of an abelian group is normal, so each subgroup gives rise to a quotient group.

2. Subgroups, quotients, and direct sums of abelian groups are abelian.

3. Every cyclic group is abelian. This is because its group operation must be commutative.

4. If a group G is abelian, the Centre of G ie. $Z(G)$ is equal to G itself.

5. If a group G is abelian (for example, all cyclic groups), then every conjugacy class in G consists of just a single element.

6.Interestingly, it is possible to find abelian subgroups of non-abelian groups. An example of this is the smallest non-abelian group, the symmetric group S_3 of order 6. All of its proper subgroups are abelian (the trivial subgroup, three subgroups of order 2 and one subgroup of order 3).

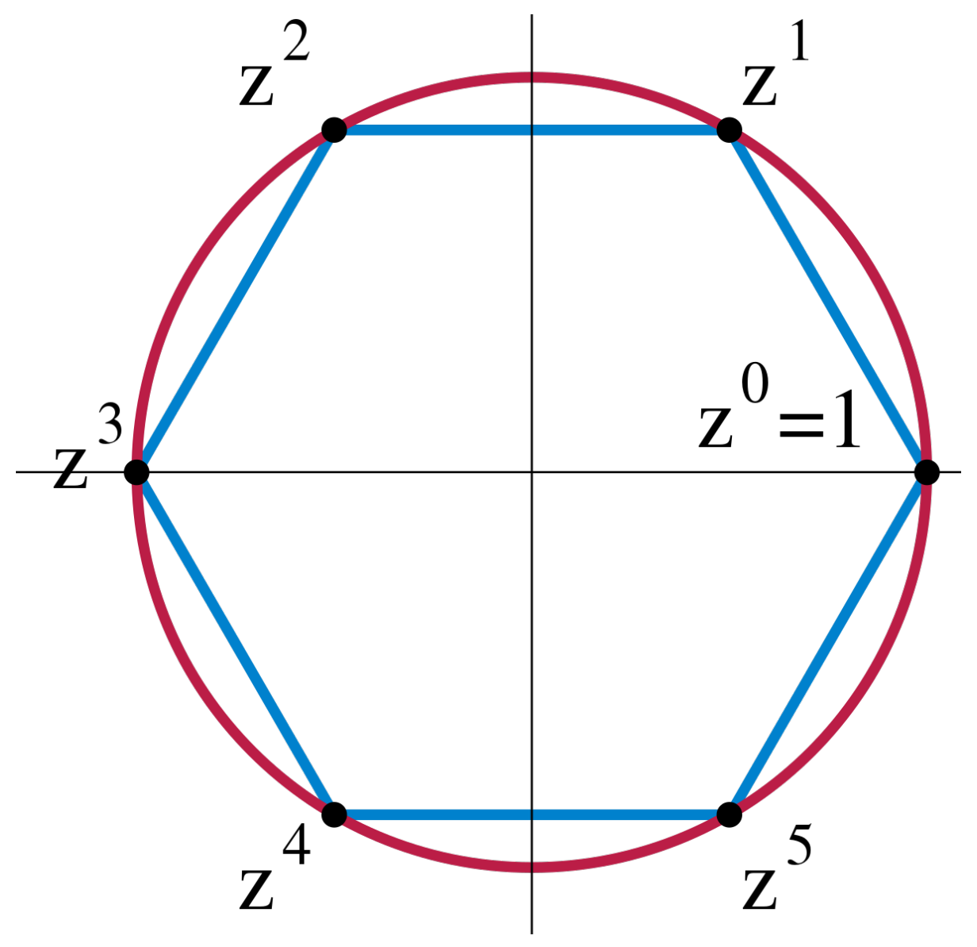

Fig. 2: Cyclic Group of 6 elements

There are also some other interesting properties relating to abelian groups. These are called "group metaproperties":

(i) Subgroup-closed group property: If G is an abelian group and H is a subgroup of G, then H is also abelian. (ie. Abelianness is subgroup-closed).

(i) Quotient-closed group property: If G is an abelian group and H is a normal subgroup of G, the quotient group G/H is also abelian. (ie. Abelianness is quotient-closed).

Examples of Abelian and Non-abelian Groups

Some abelian groups:

$(\mathbb{Z}$,+$)$ The set of all integers under the binary operation of addition.
$(\mathbb{R}^*$,x$)$ The set of all non-negative real numbers under multiplication.

Some non-abelian groups:

$(\mathbb{Z}$,+$)$ The set of all integers under the binary operation of addition.
GL$(2, \mathbb{R})$ The set of all invertible $2x2$ matrices with real entries.

Cayley Graphs for Abelian Groups

Cayley graphs give a way of encoding information about groups in a graph. Given a group with a finite generating set, we can form a Cayley Graph for that group with respect to that generating set.

Cayley's Basic Theorem states that every group can be faithfully represented as a group of permutations. The graph below represents the graph of S_4 for all the rotations of a cube. The colours are the vertices of the RGB color cube which correspond to the numbers from 0 to 7, each number representing a vertex.

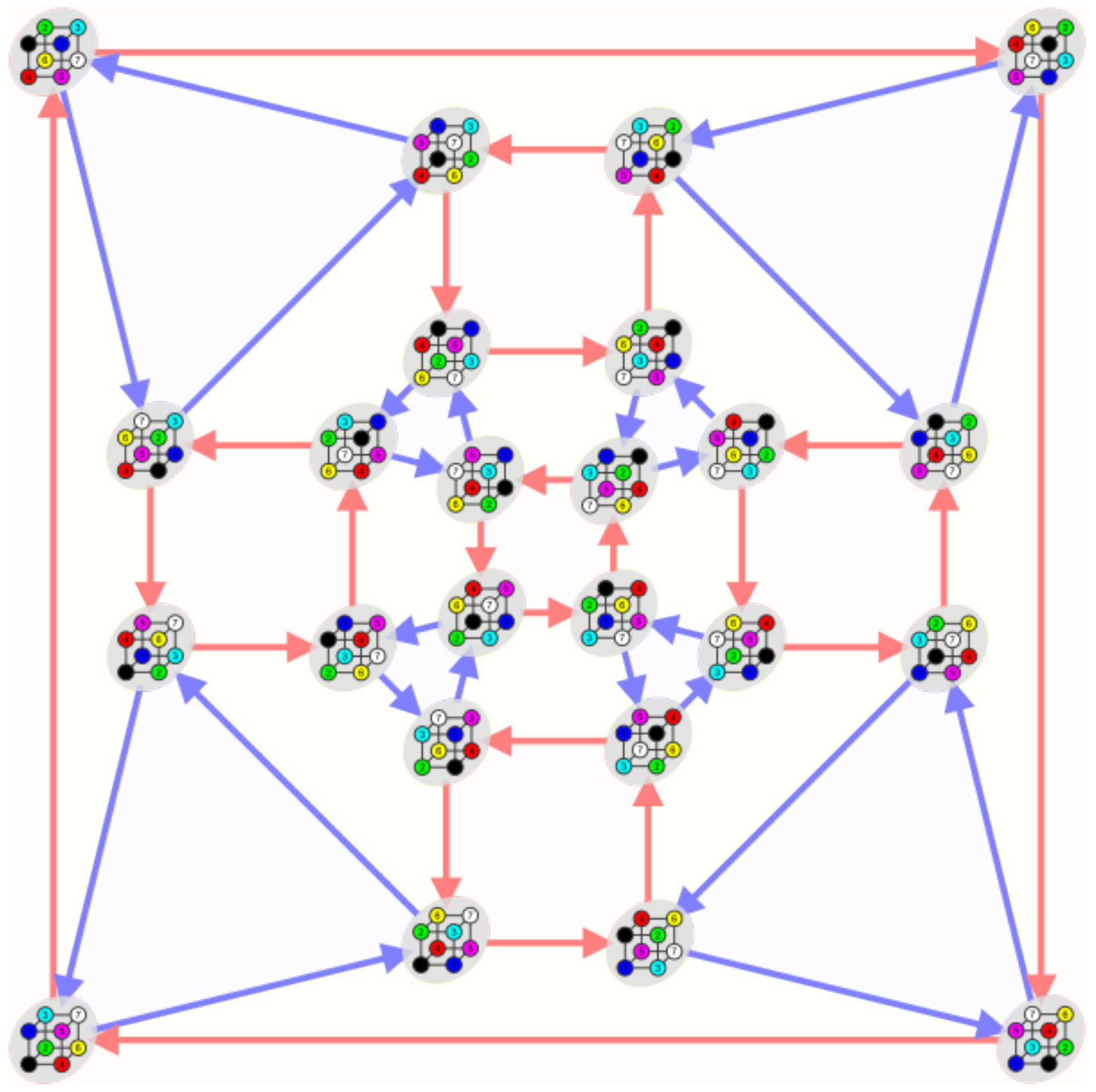

Fig. 3: Cayley graph of S_4 showing all rotations of a cube

Fundamental theorem of abelian groups

The fundamental theorem of finite abelian groups states that every finite abelian group G can be expressed as the direct sum of cyclic subgroups of prime-power order; it is also known as the basis theorem for finite abelian groups.. This is generalized by the fundamental theorem of finitely generated abelian groups, with finite groups being the special case when G has zero rank. (The rank of a group referes to the smallest cardinality of a generating set for G)

References

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