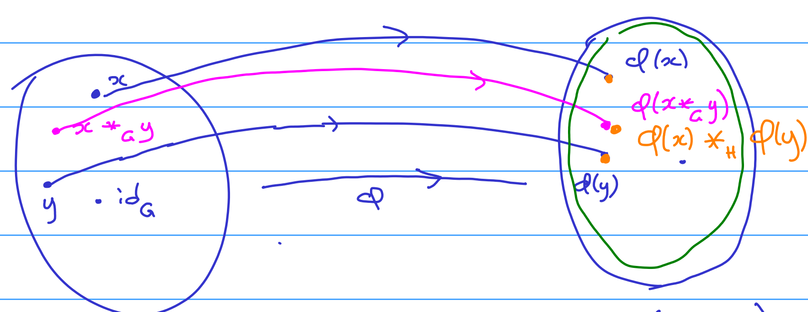
Last week: Group Homomorphisms

Functions between groups that respect the algebraic structure



$(G, *_G)$ $(H, *_H)$

$\phi(xy) = \phi(x)\phi(y)$

If $\phi(x *_G y) = \phi(x) *_H \phi(y)$ for all $x, y \in G$, then ϕ is a group homomorphism

Example: det: $GL(2,\mathbb{R}) \longrightarrow \boxed{\mathbb{R}^*}$ ← group of nonzero real numbers under multiplication

For $A, B \in GL(2,\mathbb{R})$, $\det(AB) = \det(A) \times \det(B)$

matrix product multiplication of real numbers

① $\mathbb{R}^\times \to \mathbb{R}^\times$, $x \to |x|$

non zero real numbers, $\times$

For $x, y \in \mathbb{R} \setminus \{0\}$

$$|xy| \stackrel{?}{=} |x|\,|y| \quad \checkmark \qquad \text{Yes}$$

② $GL(2,\mathbb{R}) \to GL(2,\mathbb{R})$ $\qquad A \to A^{-1}$

For $A, B \in GL(2,\mathbb{R})$

matrix product

$$(AB)^{-1} \stackrel{?}{=} A^{-1}\ B^{-1}$$

No $\to$ $(AB)^{-1} = B^{-1}A^{-1}$, which is generally not the same as $A^{-1}B^{-1}$ for 2×2 matrices

③ $(\mathbb{Z}, +) \to (\mathbb{Z}, +)$ $\qquad x \to x^2$

for $x, y \in \mathbb{Z}$

$$(x+y)^2 \stackrel{?}{=} x^2 + y^2 \qquad \text{No!}$$

Not a group homomorphism.

④ $(\mathbb{Z}, +) \longrightarrow \mathbb{R}^\times$ $\qquad x \to 2^x$

For $x, y \in \mathbb{Z}$

$$2^{x+y} = 2^x \times 2^y \quad \checkmark \qquad \text{Yes}$$

⑤ $GL(2,\mathbb{R}) \longrightarrow \underline{(\mathbb{R}, +)}$ $\qquad A \to A_{11}$

Take $A, B \in GL(2,\mathbb{R})$

$$(AB)_{11} \stackrel{?}{\longrightarrow} A_{11} + B_{11}$$

No. e.g. $\begin{pmatrix} \boxed{1} & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \boxed{2} & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \boxed{2} & 2 \\ 0 & 1 \end{pmatrix}$ $\qquad 2 \neq 1+2$