The image of a group homomorphism (again)

Let $\phi : G \to H$ be a group homomorphism with kernel N.
We saw in Week 10 that the distinct elements of $\operatorname{im}\phi$ correspond exactly to the distinct cosets of N in G: two elements of G have the same image under ϕ if and only if they belong to the same (left or right) coset of N in G.

This means that the group operation in $\operatorname{im}\phi$ (or in H) can be interpreted as a binary operation on the set of cosets of N in G, with respect to which this set is a group.

To multiply two cosets of N in G: take elements x and y respectively from each coset. Take the element $x \star_G y$ in G. The coset to which this element belongs is the product of the orginal two cosets in G/N, the quotient group of G modulo N.

The outcome does not depend on the initial choice of x and y from the two cosets.

Quotient Groups

This can be formulated without the context of a homomorphism.

Let N be a normal subgroup of a group G. Define a multiplication $\star$ on the set G/N of cosets of N in G by

$$xN \star yN = xyN.$$

- This is well-defined: suppose $xN = x_1N$ and $yN = y_1N$. We need to know $xyN = x_1y_1N$, i.e. that $(xy)^{-1}x_1y_1 \in N$. Now

  $$(xy)^{-1}x_1y_1 = y^{-1}x^{-1}x_1y_1 = \underbrace{(y^{-1}x^{-1}x_1y)}_{\in N}\underbrace{(y^{-1}y_1)}_{\in N}.$$

  So the multiplication operation on cosets is well-defined.
- This multiplication operation is associative since it comes from the associative operation of G; it has N as an identity element, and the coset xN has $x^{-1}N$ as its inverse. So G/N is a group, called the quotient group G modulo N.

Examples

1. The subgroup $5\mathbb{Z}$ of the group of integers under addition is the group consisting of all multiples of 5. The cosets of $5\mathbb{Z}$ in $\mathbb{Z}$ are the congruence classes of integers modulo 5,

   $$\bar{0} = 5\mathbb{Z},\ \bar{1} = 1 + 5\mathbb{Z},\ \bar{2} = 2 + 5\mathbb{Z},\ \bar{3} = 3 + 5\mathbb{Z},\ \bar{4} = 4 + 5\mathbb{Z}.$$

   The quotient group $\mathbb{Z}/5\mathbb{Z}$ is the additive group of integers modulo 5. The elements are the congruence classes above, and addition is defined by adding representatives of classes modulo 5.
2. The subgroup $N = \{\mathrm{id}, \mathbb{R}_{180}\}$ of D_8 is normal. There are four cosets of N in D_8. The group table for D_8/N is below

	N	$R_{90}N$	S_LN	S_MN
N	N	$R_{90}N$	S_LN	S_MN
$R_{90}N$	$R_{90}N$	N	S_MN	S_LN
S_LN	S_LN	S_MN	N	$R_{90}N$
S_MN	S_MN	S_LN	$R_{90}N$	N

Challenge 2, Week 11

The symmetric group S_n is the group of all permutations of the set $\{1, 2, \ldots, n\}$. Show that the only normal subgroup of S_n that contains the transposition $(1\ 2)$ is the full group S_n itself.

(Hint: You may use the fact that the set of all transpositions in S_n is a generating set for S_N, and your knowledge about conjugacy in symmetric groups.)