

Normal Subgroups

Definition A subgroup N of a group G is **normal** in G ($N \trianglelefteq G$) if for every $g \in G$ the left coset gN is equal to the right coset Ng .

Equivalent Definition A subgroup N is **normal** in G if $gng^{-1} \in N$ for every $n \in N$ and $g \in G$. This means N is a union of conjugacy classes of G , as N contains all G -conjugates of each of its elements.

Version 1 \implies Version 2 Suppose that $gN = Ng$ for all $g \in G$, and let $n \in N$. Then $gn \in Ng$, so $gn = n'g$ for some $n' \in N$. Thus $gng^{-1} \in N$, and the G -conjugacy class of n is contained in N .

Version 2 \implies Version 1 Suppose that N is a union of conjugacy classes, and let $g \in G$. Then, for each $n \in N$, $gng^{-1} \in N \implies gn = n'g$ for some $n' \in N$, so $gn \in Ng$. Thus $gN \subseteq Ng$, and the same argument applied to $g^{-1}ng$ shows that $gN \subseteq Ng$.

Examples of Normal Subgroups

- ▶ Every group is a normal subgroup of itself, and the trivial subgroup is a normal subgroup of every group.
- ▶ Every subgroup of an abelian group is normal (conjugacy classes in abelian groups are single elements).
- ▶ Any subgroup that is the kernel of a group homomorphism is normal.
- ▶ In D_6 the subgroup consisting of the three rotations is normal, and the subgroup consisting of any one reflection is non-normal.
- ▶ Any subgroup of index 2 in any group is normal.

Kernels of group homomorphisms are normal subgroups

Let $\phi : G \rightarrow H$ be a homomorphism of groups.

Write N for the kernel of ϕ , so $N = \ker \phi = \{x \in G : \phi(x) = \text{id}_H\}$.

Recall from Week 10 that $\ker \phi$ is a subgroup of G .

Claim $N \trianglelefteq G$.

Proof Let $n \in N$. We must show that $g \star_G n \star_G g^{-1} \in N$ for all $g \in G$. So choose $g \in G$. Then

$$\begin{aligned}\phi(g \star_G n \star_G g^{-1}) &= \phi(g) \star_H \phi(n) \star_H \phi(g^{-1}) \\ &= \phi(g) \star_H \text{id}_H \star_H \phi(g^{-1}) \\ &= \phi(g) \star_H \phi(g^{-1}) \\ &= \phi(g \star_G g^{-1}) \\ &= \phi(\text{id}_G) = \text{id}_H\end{aligned}$$

So $g \star_G n \star_G g^{-1} \in N$ for all $n \in N$ and $g \in G$, and the kernel of a group homomorphism is always a normal subgroup of the domain.

Challenge 1, Week 11

Prove that any subgroup of index 2 in a finite group is normal.

Hint Version 1 of the definition of a normal subgroup might be the easier one to use here. A subgroup of index 2 accounts for half of the elements in a finite group. The properties that we know about for left cosets (such as the left cosets determined by two different elements are either equal or disjoint) also apply to right cosets.