The kernel and image are subgroups

Let $\phi : G \to H$ be a homomorphism of groups. The kernel of ϕ is

$$\ker \phi = \{x \in G : \phi(x) = \mathrm{id}_H\}.$$

- $\mathrm{id}_G \in \ker \phi$ - from our first lecture in Week 10.
- Suppose $x, y \in \ker \phi$. Then

  $$\phi(x \star_G y) = \phi(x) \star_H \phi(y) = \mathrm{id}_H \star_H \mathrm{id}_H = \mathrm{id}_H,$$

  so $x \star_G y \in \ker \phi$ and $\ker \phi$ is closed under $\star_G$.
- Suppose $x \in \ker \phi$. Then

  $$\mathrm{id}_H = \phi(x^{-1} \star_G x) = \phi(x^{-1}) \star_H \phi(x) = \phi(x^{-1}) \star_H \mathrm{id}_H = \phi(x^{-1}).$$

  $x \in \ker \phi$

  So $x^{-1} \in \ker \phi$.

So $\ker \phi$ is a subgroup of G.

The image of ϕ is a subgroup of H (see lecture notes).

How does im ϕ resemble G?

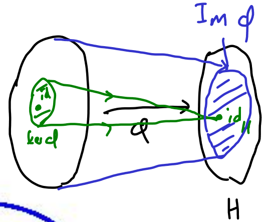
Example The modulus function from $\mathbb{C}^\times$ to $\mathbb{R}^\times_{>0}$ defined by $z \to |z|$ or $x + iy \to \sqrt{x^2 + y^2}$ is a group homomorphism. $|z_1 z_2| = |z_1|\,|z_2|$

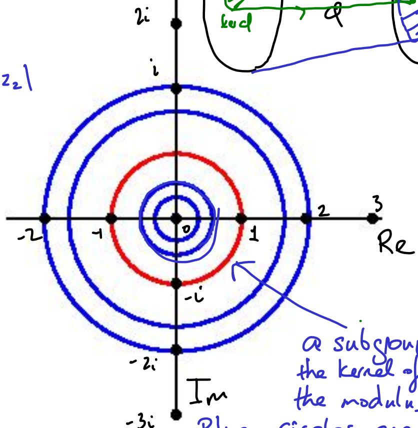
Its kernel is the unit circle.

Two elements have the same image if they are on the same circle centred at 0.

These circles are the cosets of the unit circle subgroup in $\mathbb{C}^\times$.

The image is $\mathbb{R}_{>0}$, there is one element there for every coset of the kernel in $\mathbb{C}^\times$.





a subgroup the kernel of the modulus

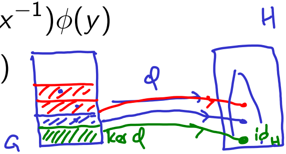
Blue circles are cosets of this subgroup

There is one element of im ϕ corresponding to every coset of ker ϕ in G.

Elements of the image correspond to cosets of the kernel

Let $\phi : G \to H$ be a group homomorphism, and let x and y be elements of G. Then

$$\begin{aligned} \phi(x) = \phi(y) &\iff \phi(x^{-1})\phi(x) = \phi(x^{-1})\phi(y) \\ &\iff \underbrace{\phi(\mathrm{id}_G)}_{=\mathrm{id}_H} = \phi(x^{-1}y) \\ &\iff x^{-1}y \in \ker\phi. \end{aligned}$$



So $\phi(x) = \phi(y)$ if and only if x and y belong to the same (left) coset of $\ker\phi$ in G.

The distinct elements of $\operatorname{im}\phi$ correspond exactly to the distinct (left) cosets of $\ker\phi$ in G. The group operation in $\operatorname{im}\phi$ determines an operation on these distinct cosets.
Finally, for every $x \in G$, the left and right cosets of $\ker\phi$ determined by x are the same set; each one of them is the set of elements of G whose image under ϕ is $\phi(x)$. This is saying that

$\ker\phi$ is a normal subgroup of G.

Challenge for Week 10

For a rational number x, $\lfloor x \rfloor$, called the floor of x, is the greatest integer that is less than or equal to x. Determine, with explanation, whether the function defined by $x \rightarrow \lfloor x \rfloor$ is a group homomorphism from $(\mathbb{Q}, +)$ to $(\mathbb{Z}, +)$.
(Note the floor function rounds every rational number down to an integer).