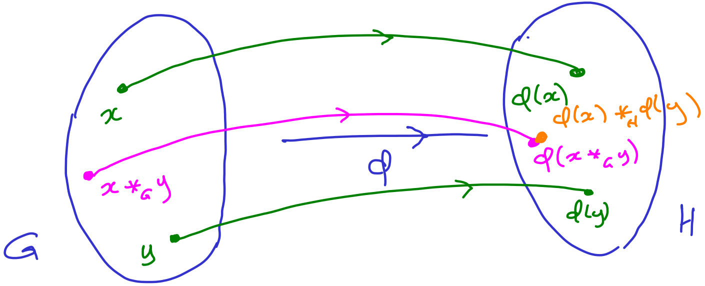
Group Homomorphisms

Definition Let G and H be groups, with operations $\star_G$ and $\star_H$ respectively. A function $\phi : G \to H$ is a **group homomorphism** if

$$\phi(x \star_G y) = \phi(x) \star_H \phi(y),$$

for all elements x, y of G.



Examples of Group Homomorphisms

1. The Determinant The function $\det : \mathrm{GL}(2, \mathbb{R}) \to \mathbb{R}^{\times}$ is a group homomorphism, since for any matrices A, B in $\mathrm{GL}(2, \mathbb{R})$,

$$\det(AB) = \det(A) \times \det(B).$$

   Note that the multiplication in the left hand side of the above equation is multiplication of 2×2 matrices and the "$\times$" on the right hand side refers to multiplication of real numbers.

2. The logarithm function Let $\mathbb{R}^{\times}_{>0}$ denote the group of all positive real numbers under multiplication, and let $x, y \in \mathbb{R}^{\times}_{>0}$. Then $\log(x)$, $\log(y)$ and $\log(x + y)$ are real numbers, and

$$\log(xy) = \log(x) + \log(y).$$

   The function log is a group homomorphism from $\mathbb{R}^{\times}_{>0}$ to $(\mathbb{R}, +)$, the group of all real numbers under addition. (Note that the choice of base of log does not matter here).

Lemma: $\phi(\mathrm{id}_G) = \mathrm{id}_H$

Let $\phi : G \to H$ be a group homomorphism, and let $x \in G$. Then

$$\phi(x \star_G \mathrm{id}_G) = \phi(x) \star_H \phi(\mathrm{id}_G).$$

Also $\phi(x \star_G \mathrm{id}_G)(x \star_G \mathrm{id}_G) = \phi(x)$.
In H we have

$$\begin{aligned} \phi(x) \star_H \phi(\mathrm{id}_G) &= \phi(x) \\ \implies \phi(x)^{-1} \star_H \phi(x) \star_H \phi(\mathrm{id}_G) &= \phi(x)^{-1} \star_H \phi(x) \\ \implies \phi(\mathrm{id}_G) &= \mathrm{id}_H\,. \end{aligned}$$

Definition The **kernel** of a group homomorphism $\phi : G \to H$ is the subset of G consisting of all those elements whose image under ϕ is id_H.
The **image** of ϕ is the subset of H consisting of all $\phi(x)$, where $x \in G$.

Examples again

1. det : $\mathrm{GL}(2, \mathbb{R}) \to \mathbb{R}^\times$
   The kernel is $\{A \in \mathrm{GL}(2, \mathbb{R}) : \det A = 1\}$. This is the special linear group $\mathrm{SL}(2, \mathbb{R})$. It is a subgroup of $\mathrm{GL}(2, \mathbb{R})$.
   The image of det includes all elements of $\mathbb{R}^\times$.
2. The kernel of $\log : \mathbb{R}^\times_{>0} \to (\mathbb{R}, +)$ is

$$\{x \in \mathbb{R}_{>0} : \log(x) = 0\} = \{1\}.$$

   The kernel consists only of the identity element of $\mathbb{R}^\times_{>0}$. The image of log is the entire group $(\mathbb{R}, +)$.

In the next instalment we will see that the kernel and image of a group homomorphism $\phi : G \to H$ are subgroups of G and H respectively.